Nig. J. Pure & Appl. Sci. Vol. 31 (Issue 2, 2018)
ISSN 0794-0378
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www.njpas.com.ng

doi: http://dx.doi.org/10.19240/njpas.2018.A24

VARIABLE CONTROL CHARTS BASED ON PERCENTILES OF EXPONENTIATED LOMAX DISTRIBUTION

Adewara Johnson Ademola[1*] and Aako Olubisi Lawrence[2]

[1] Distance Learning Institute, University of Lagos, Akoka, Lagos, Nigeria.

[2] Departments of Mathematics & Statistics, the Federal Polytechnic, Ilaro, Ogun State, Nigeria.

Abstract

Statistical quality control relies heavily on the goodness of control chart limits. The more accurate those limits are, the more likely are to detect whether a process is in control. Various procedures have been developed to compute good control limits. This paper proposes variable control charts based on the percentiles of Exponentiated Lomax distribution (ELD) for mean, range and standard deviation. The percentiles of the distribution of mean, range and standard distribution were developed and are used to construct the control limits. The coverage probability of the control charts of ELD was compared with that of traditional shewhart control chart. The result shows that ELD performs better than the traditional shewhart control chart.

Key words: Skewed Distribution, Variable Control Charts, Percentiles, Exponentiated Lomax

Introduction

Life data is refers to as measurements of product life. Reliability Life time data generally contain the failure times of sample products or number of failures experienced in a given time (Kantam and Ravi Kumar, 2013). Product life can be measured in hours, miles, cycles or any other metric that applies to the period of successful operation of a particular product. Since time is a common measure of life, life data points are often called "times-to-failure" and product life will be described in terms of time in this paper. In quality control studies, data is always in small samples only which may not be normally distributed. The well-known Shewart control charts developed under the assumption that the quality characteristic follows a normal distribution. When the underlying distribution is skewed, there are potential problems, namely, the false alarm rates and detection power of an out-of-control condition often substantially differ from what we expect under the normal case (Mahoney, 1998; Tadikamalla, Banciu, & Popescu, 2008). Therefore if a quality variate is not normal there is a need to develop a separate

Corresponding Author: Adewara Johnson Ademola, Distance Learning Institute, University of Lagos, Akoka, Lagos, Nigeria.
Email: adewaraja@gmail.com, aako.olubisi@yahoo.com

procedure for the construction of control limits as the central limit theorem cannot be made use of, because central limit theorem can be used when the sample size is large and it gives only asymptotic normality for any statistic.

Many authors have attempted the development of statistical quality control methods for skewed distributions. Some of them are Edgeman (1989) -Inverse Gaussian Distribution, Gonzalez and Viles (2000) -Gamma Distribution, Kantam and Sriram (2001) -Gamma Distribution, Chan and Cui (2003) Several other method have been used to developed control charts for skewed data such as Kantam *et al*. (2006) -Log logistic Distribution, Betul and Yaziki (2006) –Burr Distribution, Subba Rao and Kantam (2008) -Double exponential distribution, Kantam and Srinivasa Rao (2010) -control charts for process variate, Srinivasa Rao and Sarath Babu (2012) -Linear failure rate distribution, Srinivasa Rao and Kantam (2012) -Half logistic distribution, Srinivasa Rao, Durgamamba and Subba Rao (2014) presented the construction of quality control charts when the process variate is assumed to follow size biased Lomax distribution, Rezac, Lio and Jiang (2015) developed Burr Type-XII Percentile Control Charts, Boyapati, Nasiru, and Lakszhmi (2015) developed variable control charts based on percentiles of the new Weibull-Pareto distribution but the problem of exponential lomax to fit skewed data in quality control has not been fixed.

This paper focuses on developing control charts using Exponential Lomax distribution to develop control charts in monitoring and controlled skewed data.

Exponentiated Lomax distribution

Exponentiated Lomax is another approach to skewed distribution that was not paid much attention with respect to development of control charts. At the same time it is one of the probability models applicable for life testing and reliability studies. If a lifetime data is a quality data, the development of control charts for the same is desirable for the use by practitioners. Exponentiated Lomax distribution (ELD) is a generalization of Lomax distribution by powering a positive real number α to the cumulative distribution function.

Let X be a random variable from an Exponentiated Lomax distribution with its cumulative distribution function (cdf) given by

$$F(x) = \left[1 - (1+\lambda x)^{-\theta}\right]^{\alpha}; \quad x > 0,\ \theta, \alpha, \lambda > 0 \qquad (1.1)$$

The probability density function (pdf) corresponding to (1.1) is

$$f(x) = \alpha\theta\lambda\left[1 - (1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}; \quad x > 0,\ \theta, \alpha, \lambda > 0 \qquad (1.2)$$

when λ = 1, the pdf reduces to exponentiated pareto distribution (θ, α), when $\alpha = \lambda = 1$, it reduces to standard Lomax distribution (θ).

The distributional properties are:

$$E(X) = \frac{\alpha}{\lambda}\left[B\left(1 - \frac{1}{\theta}, \alpha\right) - B(1, \alpha)\right] \qquad (1.3)$$

$$Var(X) = \left(\frac{\alpha}{\lambda}\right)^2\left[\frac{1}{\alpha}B\left(1 - \frac{2}{\theta}, \alpha\right) - B^2\left(1 - \frac{1}{\theta}, \alpha\right)\right]; \quad \theta > 2 \qquad (1.4)$$

The quantiles is given by

$$X_p = \frac{1}{\lambda}\left[\frac{1}{\left(1 - p^{1/\alpha}\right)^{1/\theta}} - 1\right] \qquad (1.5)$$

The pdf of the largest order statistic $X_{(n)}$ is given by

$$X_{(n)} = \alpha\theta\lambda\left[1 - (1+\lambda x)^{-\theta}\right]^{\alpha n-1}(1+\lambda x)^{-(\theta+1)} \qquad (1.6)$$

The pdf of the smallest order statistic $X_{(1)}$ is given by

$$X_{(1)} = n\alpha\theta\lambda\left[1 - (1+\lambda x)^{-\theta}\right]^{\alpha-1}(1+\lambda x)^{-(\theta+1)}\left[1 - \left[1 - (1+\lambda x)^{-\theta}\right]^{\alpha}\right]^{n-1} \qquad (1.7)$$

The other distributional properties are thoroughly discussed by Abdul-Moniem and Abdel-Hameed (2012) and Salem (2014).

Methods

The data for this study was simulated from a random sample of size n with $x_1, x_2, .., x_n$ drawn from Exponentiated Lomax Distribution with $\alpha = 2.0, \lambda = 0.3$ and $\theta = 3.0$ using the methods adopted by Srinivasal *et al*. (2014). This is considered as a subgroup of an industrial process data with a targeted population average, under repeated sampling. The statistic $\bar{x}$ gives whether the process average is around the targeted mean or not. The control limits to which the $\bar{x}$ falls has to be determined. Hence, the concept of 3σ limits is taken as the 'most probable' limits. It is well known that 3σ limits of normal distribution include 99.73% of probability.

Results and Discussion

The results for the Mean, Range and Standard deviation for exponentiated lomax distribution are presented in the table 1.

Table 1: Percentiles of Mean in ELD

N	0.99865	0.9950	0.99	0.975	0.95	0.05	0.025	0.01	0.005	0.00135
2	19.6451	13.3550	10.7559	7.9266	6.1616	0.5479	0.4270	0.3152	0.2542	0.1730
3	13.1624	9.4309	7.8159	5.9926	4.8116	0.6300	0.5180	0.4104	0.3488	0.2621
4	11.9924	8.8068	7.4009	5.7892	4.7285	0.7803	0.6629	0.5473	0.4795	0.3813
5	7.5939	5.7638	4.9303	3.9506	3.2892	0.6466	0.5598	0.4730	0.4213	0.3453
6	6.2047	4.7945	4.1419	3.3652	2.8340	0.6299	0.5532	0.4757	0.4291	0.3598
7	3.8564	3.0457	2.6619	2.1966	1.8725	0.4568	0.4045	0.3512	0.3190	0.2707
8	3.7755	3.0026	2.6350	2.1877	1.8749	0.4910	0.4385	0.3850	0.3524	0.3032
9	2.9227	2.3546	2.0810	1.7449	1.5075	0.4260	0.3835	0.3398	0.3131	0.2725
10	3.1132	2.5199	2.2338	1.8817	1.6327	0.4917	0.4462	0.3993	0.3705	0.3266

The percentiles in the above table are used in the following manner to get the control limits for sample mean. From the distribution of $\bar{x}$, the upper and lower control limits is obtained thus:

$$P(Z0.00135 \leq \bar{x} \leq Z0.99865) = 0.9973 \quad (2.2)$$

The $\bar{x}$ of sampling distribution when $\alpha = 2.0$, $\lambda = 0.3$ and $\theta = 3.0$ is 2.6667 for ELD. This is obtain from equation (2.2) above over repeated sampling. The values in equation 2.3 is obtained for the *ith* subgroup mean

$$P(Z_{0.00135}\frac{\bar{\bar{x}}}{2.6667} \leq \bar{x} \leq Z_{0.998\ 65}\frac{\bar{\bar{x}}}{2.6667}) = 0.9973 \quad (2.3)$$

Or

$$P(A_{2p}^{*} \times \bar{\bar{x}} \leq \bar{x}_i \leq A_{2p}^{**} \times \bar{\bar{x}}) = 0.9973 \quad (2.4)$$

where $\bar{\bar{x}}$ is grand mean, $\bar{x}_i$ is *ith* subgroup mean, $A_{2p}^{*} = \frac{Z_{0.00135}}{2.6667}$, $A_{2p}^{**} = \frac{Z_{0.9\ 9\ 8\ 6\ 5}}{2.6667}$. Also, A_{2p}^{*}, A_{2p}^{**} are the percentile constants of $\bar{x}$ chart for ELD as given in Table 2.

Table 2: Percentile constants of Mean-chart

n	A_{2p}^{*}	A_{2p}^{**}
2	0.0649	7.3668
3	0.0983	4.9358
4	0.1430	4.4971
5	0.1295	2.8477
6	0.1349	2.3267
7	0.1015	1.4461
8	0.1137	1.4158
9	0.1022	1.0960
10	0.1225	1.1674

The range chart percentiles was developed for the exponentiated lomax distribution, the limits of the sampling distribution for the sample range in ELD with given probability content of these limits are 0.9973. This is of the form: $P(L \le R \le U) = 0.9973$ (2.5)

where R is the range and n is the sample of size. The simulation was performed and the percentile is given in Table 3.

Table 3: Percentiles of Range in ELD

N	0.99865	0.9950	0.99	0.975	0.95	0.05	0.025	0.01	0.005	0.00135
2	14.0932	8.6993	6.6196	4.4766	3.2164	0.0411	0.0212	0.0090	0.0047	0.0014
3	19.9184	12.5120	9.6534	6.7030	4.9615	0.2242	0.1561	0.0994	0.0715	0.0391
4	30.8236	19.2190	14.8079	10.3069	7.6798	0.5359	0.4105	0.2972	0.2363	0.1568
5	36.3986	22.8193	17.6558	12.3848	9.3060	0.8254	0.6612	0.5079	0.4225	0.3057
6	43.4922	27.2799	21.1327	14.8708	11.2206	1.1482	0.9440	0.7501	0.6401	0.4861
7	47.6453	29.9467	23.2381	16.4056	12.4231	1.3959	1.1658	0.9453	0.8189	0.6397
8	47.6652	30.1144	23.4399	16.6245	12.6412	1.5181	1.2802	1.0510	0.9188	0.7299
9	47.2963	30.0895	23.5140	16.7746	12.8202	1.6531	1.4077	1.1698	1.0318	0.8333
10	56.1102	35.5110	27.6879	19.7080	15.0485	2.0162	1.7302	1.4522	1.2902	1.0564

The percentiles from the above table are used to obtain the control limits for sample range from R distribution given $P(Z0.00135 \le R \le Z0.99865) = 0.9973$ (2.6)

From equation (2.6), for the *ith* subgroup range is

$$P(Z_{0.00135}\frac{\bar{R}}{X_{(n)} - X_{(1)}} \le R_i \le Z_{0.998\,65}\frac{\bar{R}}{X_{(n)} - X_{(1)}}) = 0.9973 \quad (2.7)$$

Or $P(D_{3p}^{*}\bar{R} \le R_i \le D_{3p}^{**}\bar{R}) = 0.9973$ (2.8)

where $\bar{R}$ is mean of ranges, R*i* is *ith* subgroup range. $X_{(1)}$ is calculated by $F(x) = \frac{1}{n+1}$ and $X_{(n)}$ by $F(x) = \frac{n}{n+1}$. Thus, $D_{3p}^{*} = \frac{Z_{0.00135}}{X_{(n)} - X_{(1)}}$, $D_{3p}^{**} = \frac{Z_{0.9\,9\,8\,6\,5}}{X_{(n)} - X_{(1)}}$ are the percentile constants of R chart for ELD.

The percentiles constant is given in Table 4 below.

Table 4: Percentile constants of Range-chart

N	D_{3p}^{*}	D_{3p}^{**}
2	0.001	9.8961
3	0.0169	8.6055
4	0.0524	10.3046
5	0.0861	10.2572
6	0.1207	10.7965
7	0.1437	10.6997
8	0.1509	9.8563
9	0.1607	9.1193
10	0.2037	10.8188

The control limits for the sample standard deviation in ELD such that its probability limits is 0.9973. This implies that the values of lower and the upper control limits is of the form

$$P(L \le s \le U) = 0.9973 \quad (2.9)$$

Also, *s* is the standard deviation and n the sample size. The simulation was performed for percentiles is given in Table 5.

Table 5: Percentiles of Standard deviation in ELD

N	0.99865	0.9950	0.99	0.975	0.95	0.05	0.025	0.01	0.005	0.00135
2	19.9315	12.3031	9.3618	6.3311	4.5487	0.0581	0.0300	0.0127	0.0067	0.0020
3	48.6372	29.9050	22.8248	15.6338	11.4592	0.5006	0.3489	0.2225	0.1604	0.0881
4	83.0751	50.8738	38.8397	26.7196	19.7399	1.3247	1.0134	0.7328	0.5823	0.3861
5	80.4309	50.6506	39.2758	27.6241	20.7942	1.8398	1.4708	1.1267	0.9351	0.6738
6	118.7780	74.1762	57.3327	40.2280	30.2889	3.0660	2.5194	2.0009	1.7067	1.2954
7	127.6788	80.4062	62.4556	44.1483	33.4623	3.7762	3.1542	2.5580	2.2161	1.7314
8	113.5955	72.7422	57.0088	40.7867	31.2102	3.8428	3.2410	2.6602	2.3247	1.8454
9	114.7688	74.2830	58.5663	42.2607	32.5729	4.3916	3.7480	3.1223	2.7582	2.2334
10	121.7931	79.1853	62.5995	45.3554	35.0869	5.0059	4.3059	3.6223	3.2229	2.6439

The control limits for the sample standard deviation in ELD such that its probability limits is 0.9973. This implies that the values of s is of the form

P (Z0.00135 ≤ s ≤ Z0.99865) = 0.9973 (2.10)

But standard deviation of sampling distribution when α = 2.0, λ = 0.3 and θ =3.0 is 3.7417 for ELD. From equation (2.10), for the ith subgroup standard deviation is

$$P\left(Z_{0.00135}\frac{\bar{s}}{3.7417} \leq s_i \leq Z_{0.998\ 65}\frac{\bar{s}}{3.7417}\right) = 0.9973 \quad (2.11)$$

Or $P\left(B_{3p}^{*} \times \bar{s} \leq s_i \leq B_{3p}^{**} \times \bar{s}\right) = 0.9973$ (2.12)

where $\bar{s}$ is grand mean, s_i is ith subgroup mean,

$B_{3p}^{*} = \frac{Z_{0.00135}}{3.7417}$, $B_{3p}^{**} = \frac{Z_{0.9\ 9\ 8\ 6\ 5}}{3.7417}$. Thus, B_{3p}^{*}, B_{3p}^{**} are the percentile constants of SD- chart for ELD are given in Table 6.

Table 6: Percentile constants of SD-chart

n	B_{3p}^{*}	B_{3p}^{**}
2	5e-04	5.3269
3	0.0235	12.9987
4	0.1032	22.2025
5	0.1801	21.4958
6	0.3462	31.7444
7	0.4627	34.1232
8	0.4932	30.3593
9	0.5969	30.6729
10	0.7066	32.5502

Table 7: Coverage Probabilities of Mean-chart

	Shewhart limits			Percentile limits ELD		
n	$\bar{\bar{X}} - A_2\bar{R}$	$\bar{\bar{X}} + A_2\bar{R}$	Coverage probabmility	$A_{2p}^{*} \times \bar{\bar{x}}$	$A_{2p}^{**} \times \bar{\bar{x}}$	Coverage probability
2	-0.2054	0.6658	**0.965**	0.0149	1.6957	**0.996**
3	-0.0140	0.5084	**0.929**	0.0243	1.2203	**0.996**
4	0.0275	0.4557	**0.924**	0.0340	1.0703	**0.997**
5	0.0500	0.4162	**0.928**	0.0302	0.6637	**0.984**
6	0.0696	0.4047	**0.903**	0.0320	0.5518	**0.972**
7	0.0745	0.3983	**0.924**	0.0240	0.3419	**0.872**
8	0.0984	0.3766	**0.910**	0.0277	0.3453	**0.882**
9	0.0970	0.3718	**0.912**	0.0240	0.2569	**0.696**
10	0.1138	0.3700	**0.882**	0.0294	0.2804	**0.749**

Table 8: Coverage Probabilities of Range-chart

	Shewhart limits			Percentile limits ELD		
n	$D_3\bar{R}$	$D_4\bar{R}$	Coverage probability	$D^*_{3p}\bar{R}$	$D^{**}_{3p}\bar{R}$	Coverage probability
2	0	0.7516	**0.945**	0.00023	2.2745	**0.998**
3	0	0.9114	**0.932**	0.0064	3.2796	**0.997**
4	0	1.0294	**0.932**	0.0237	4.6522	**0.999**
5	0	1.0577	**0.916**	0.0432	5.1420	**0.988**
6	0	1.1555	**0.913**	0.0697	6.2377	**1.000**
7	0.0511	1.2254	**0.908**	0.0917	6.8289	**0.995**
8	0.1003	1.3320	**0.918**	0.1081	7.0586	**0.995**
9	0.1285	1,2995	**0.903**	0.1147	6.5114	**0.996**
10	0.1800	1.3832	**0.892**	0.1583	8.4069	**0.997**

Table 9: Coverage Probabilities of SD-chart

	Shewhart limits			Percentile limits ELD		
n	$B_3\ \bar{s}$	$B_4\ \bar{s}$	Coverage probability	B^*_{3p}	B^{**}_{3p}	Coverage probability
2	0.0033	0.6696	**0.946**	8.6643e-5	0.8657	**0.982**
3	0.0082	0.6058	**0.936**	0.00482	2.6605	**0.994**
4	0.00204	0.5147	**0.948**	0.0215	4.6340	**0.987**
5	0.0335	0.4773	**0.905**	0.0459	5.8465	**0.953**
6	0.0490	0.4745	**0.896**	0.0771	7.0712	**0.879**
7	0.0604	0.4669	**0.884**	0.1075	7.9261	**0.787**
8	0.0745	0.4793	**0.867**	0.1225	7.5396	**0.756**
9	0.0782	0.4298	**0.860**	0.1409	7.2429	**0.653**
10	0.0913	0.4466	**0.838**	0.1743	8.0306	**0.574**

The control chart for the statistics i.e the mean, range and standard deviation developed in Tables 2, 4 and 6 is based on the population described by ELD. The result shows that the data use for this paper follows ELD. Therefore the power of the control limits can be assessed for ELD data using Shewhart limits. Also the comparative analysis study of simulating random samples of size n=2,…,10 for ELD and the calculated control limits using the constants of Tables 2, 4 and 6 for mean, range and standard deviation was developed in succession. The number of statistic values that have fallen within the respective control limits is called ELD coverage probability. Similar count for control limits using Shewhart constants available in quality control was also calculated for the mean, range and standard deviation. The coverage probabilities under the two schemes namely the ELD and Shewhart limits are presented in the following Tables 7, 8 and 9.The results shows that ELD performs better than the Shewart control chart by reflection some of the variation on the coverage probability.

Conclusions

The Tables 7, 8 and 9 show that for ELD if the Shewhart limits are used in decision making, it would result in less confidence coefficient about the decision of process variation for mean, range and standard deviation charts especially when the number of samples is less than or equal to 5. Hence if a data is confirmed to follow ELD, the usage of Shewhart constants in all the above charts is not advisable hence, the use of ELD constants is preferable in the evaluation and the performance of statistical quality control.

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