

## Analysis of General Phase Type Inter-Arrival and Service Time Distributions in Solving Queueing Problems Using Neuts' –Matrix Method

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### ABSTRACT

Queues form an indispensable part of our everyday lives. They occur whenever there is competition for limited resources. For example, we line up in queues outside doctors' offices and supermarket checkout counters; airplanes queue along airport runways. Therefore, our ability to model and analyse systems of queues helps to minimize their inconveniences and maximize the use of the limited resources. In this paper, a queueing system comprising of two- phase hyper-exponential inter-arrival time, Erlang-3 distributed service time, single server and infinite number of arrivals has been investigated in order to provide some insight into the analysis of phase type distributions queues. This is carried out by using procedure for solving phase-type queueing systems by means of the matrix-geometric approach namely, construction of the block sub-matrices, forming the Neuts  $R$  matrix, solving the boundary equations and generating the successive components of the solution. Performance Measures for phase type Queueing system, Ph/Ph/1, are obtained as follows, the probability that there are  $k$  customers present in the queueing system, for  $k \geq 1$ , is  $p_k = \|\pi_k\|_1 = \|\pi_1 R^{k-1}\|_1$ , the probability that the system is empty is given by  $p_0 = \|\pi_0\|_1$ , The probability that the system is busy is given by  $1 - p_0 = 1 - \|\pi_0\|_1 = 1 - \rho$ , The probability that there  $k$  or more customers present is,  $\text{Prob} [N \geq k] = \sum_{j=k}^{\infty} \|\pi_j\|_1 = \|\pi_1 \sum_{j=k}^{\infty} R^{j-1}\|_1 = \|\pi_1 R^{k-1} \sum_{j=0}^{\infty} R^j\|_1 = \|\pi_1 R^{k-1} (1 - R)^{-1}\|_1$ . By Little's law and PASTA property, the average number of customers in phase type Ph/Ph/1, Mean number of customers waiting in the queue, the average response time, the average time spent waiting in the queue were obtained. The practical examples were then considered for certain given values of  $\lambda$ ,  $\mu$ , and  $\gamma$  to obtained  $\pi_i = (0.013 \quad 0.0356 \quad 0.0336) \forall i = 0, 1, 2$ .

**Keywords:** Birth –Death Matrix, Infinitesimal Generator, Quasi Birth- Death Process, Neut's Rate Matrix

### Introduction

The probability law used to model an inter-arrival or service time distribution under Markovian queue is the exponential distribution. This gave rise to birth – death processes (Kleinrock, 1975). In these systems, transitions from any state are to adjacent states only and the resulting structure of the transition matrix is tri-diagonal. But sometimes the exponential distribution is simply not adequate.

Queueing system with phase type arrival or service mechanisms give rise to transition matrices that are block tri-diagonal and are referred to as quasi – birth-death (QBD) processes. In a tri – diagonal matrix of simple birth-death process, elements below the diagonal represents service completions (departures from the system) and elements above the diagonal describe customer arrival. In (QBD)

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quasi-birth-death process, sub-diagonal blocks represent a more complex departure process and super-diagonal blocks represent a more complex arrival. The past mathematical techniques used to solve such queueing systems depended upon the use of Z – transform but with the invention of Neut's geometric approach, the queueing system can be solved recursively without stress with the knowledge of matrix, partial fraction and Calculus. Stochastic application of queueing theory which involves various probability distribution and its application is discussed in Law (2000) and the new Convergence Results on Functional Techniques for the Numerical Solution of M/G/1 Type Markov Chains is established. Agboola (2007) studied a single server queue where the inter arrival time is Markovian time and service time is general. This model generalizes the well-known M/G/1 queue. The waiting time process is directly analysed by solving the Lindley's equation using transform method. The Laplace Stieltjes transforms (LST) of the steady state waiting time and queue length distribution are both derived, and used to obtain recursive equations for the calculation of moments. G/M/1 and G/M/K are discussed in William (2009), where the service process has exponential distribution with mean service time  $\frac{1}{\mu}$ . i.e.  $B(x) = 1 - \exp(-\mu x)$ ,  $x \geq 0$ , while the arrival process is general with mean inter arrival time equal to  $\frac{1}{\lambda}$ . The embedded time instants are precisely the instants of customer arrivals, since the elapsed inter arrival time at these moments is known as zero. This allows us to form a transition probability matrix and to compute the distribution of customers as seen by an arriving customer. Agboola (2011) obtained the probability of an empty system and the probability of being in phase  $i$  when there are 3 customers in the system, while Charan *et al.* (2012) investigated the single server queueing system wherein the arrival of the units follows a Poisson process with varying arrival rates in different states. The server may take a vacation of a fixed duration or may continue to be available

in the system for next service. Probability generating function of the units present in the system and various performance indices such as expected number of units in the queue and in the system, average waiting time, e.t.c. are obtained. Michiel *et al.* (2017) analysed a non-classical discrete time queueing model where customers demand variable amounts of work from a server that is able to perform this work at a varying rate. The service demands of the customers are integer numbers of work units. They are assumed to be independent and identically distributed (i.i.d) random variables. An analysis method, which is based on complex contour integration, is presented and expressions are obtained for the probability generating functions, the mean values and the tail probabilities of the customer delay and the system steady state with numerical example illustration. Charan *et al.* (2019) considered a single server queueing system with batch arrival of the units. The provision of optimal service after availing essential service is available in the system, and it is assumed that the server may break down during any stage of the service of the units and provides the repair facility immediately. Phase type distribution allows us to consider more general situations and were seen to include such common distributions as Erlang and hyper-geometric distribution.

#### NOTATION

The general phase type inter-arrival and service time distributions queueing system is developed based on the following notation

$\pi$	Probability vectors
$Q$	Infinitesimal Generator Matrix
$\lambda$	Arrival rate
$r\mu$	Service rate
$r$	Number of service phases
$k$	Number of customers in the system,
$i$	The current phase of service
$R$	The Neut's rate matrix

**Materials and Methods**

The state of this queueing system is describe by three parameters  $(k, a, s)$ , the first,  $k$ , to denote the number of customers actually present, the second,  $a$ , to describe the arrival phase of the “arriving” customer, and the last,  $s$ , to indicate the current phase of service. In drawing the state transition rate diagram, we shall present the states in rows (or levels) according to the number of customers present.

Within each level  $k$ , states are ordered first according to the arrival phase of the hyper-exponential distribution and second according to the phase of service. In other words, the states  $(k, a, s)$  are ordered lexicographically. Since the state transition rate diagram can become rather complicated, we show the states and the transitions generated by the arrival process in one diagram i.e., figure 3, and in a separate diagram, we show the states and the transitions generated by the service process. i.e., figure 2. Obviously, the complete diagram is obtained by superimposing both sets of transitions onto the same state space. The first three levels of states and transitions generated by the service process are shown in Figure 1.

**THE  $H_2/E_3/1$  QUEUE AND GENERAL  $ph/ph/1$  QUEUES**

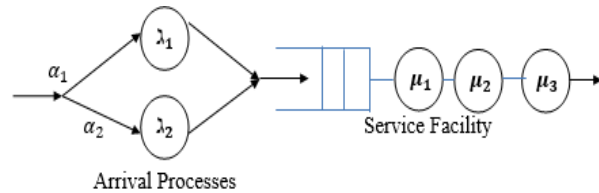
The analysis of  $H_2/E_3/1$  is necessary in order to provide insight into the analysis of more complex  $ph/ph/1$  queues. The arrival process of  $H_2/E_3/1$  is a two-phase hyper – exponential distribution while the service process is an Erlang – 3 distribution. Transition at rate  $\mu_1$  move the system from states  $(k, a, 1)$  to state  $(k, a, 2)$ ;  $\mu_2$  move the system from  $(k, a, 2)$  to  $(k, a, 3)$  while  $\mu_3$  move the system from  $(k, a, 3)$  to state  $(k - 1, a, 1)$  for  $(k = 1, 2, \dots)$ . It is only when the last service phase is completed that a departure actually occurs.

From states  $(k, 1, s)$  transitions occur at rate  $\lambda_1$ . With probability  $\alpha_1$  the next customer to begin its arrival process enter phase 1, so that with rate  $\alpha_1\lambda_1$ ,

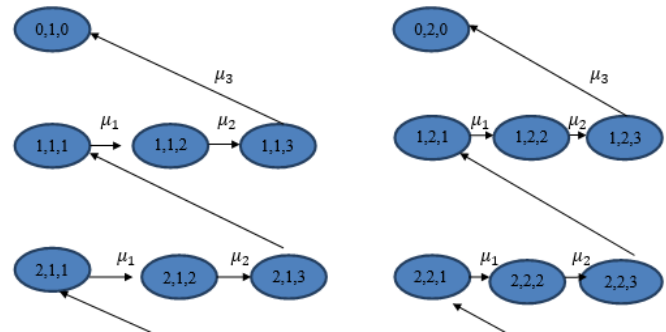
the system moves from state  $(k, 1, s)$  to  $(k + 1, 1, s)$ ; with probability  $\alpha_2$  the customer beginning its arrival phase chooses phase 2 and so at rate  $\alpha_2\lambda_1$  the system moves from  $(k, 1, s)$  to  $(k + 1, 2, s)$ .

A similar set of transitions at rate  $\alpha_1\lambda_2$  from states  $(k, 2, s)$  to  $(k + 1, 1, s)$  and transitions at rate  $\alpha_2\lambda_2$  from states  $(k, 2, s)$  to  $(k + 1, 2, s)$ .

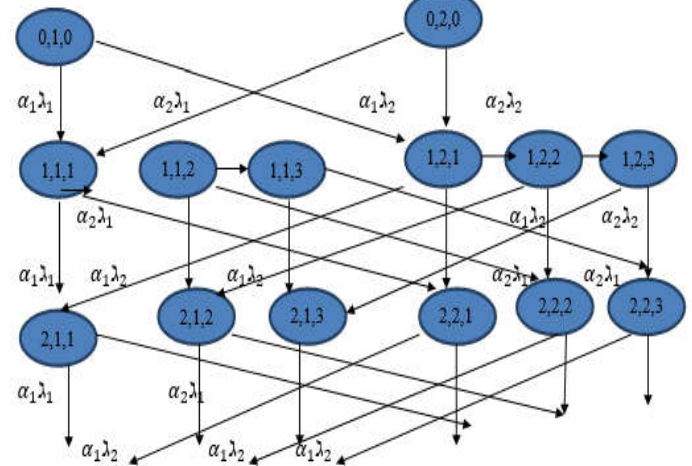
This queueing system is shown below:



**Figure 1:** The  $H_2/E_3/1$  QUEUE. (Agboola, 2007)



**Figure 2:** Transition diagram generated by the service process for the  $H_2/E_3/1$ .



**Figure 3:** Transition generated by the inter-arrival process for the  $H_2/E_3/1$  Queue

The few rows of transition matrix are as follow:

$-\lambda_1$ 0	$\alpha_1\lambda_1$ 0 0 $\alpha_2\lambda_1$ 0 0	0 0 0 0 0 0
0 $-\lambda_2$	$\alpha_1\lambda_2$ 0 0 $\alpha_2\lambda_2$ 0 0	0 0 0 0 0 0
0 0	* $\mu_1$ 0 0 0 0	$\alpha_1\lambda_1$ $\mu_1$ 0 0 0 0
0 0	0 * $\mu_2$ 0 0 0	0 $\alpha_1\lambda_1$ $\mu_2$ 0 0 0
$\mu_3$ 0	0 0 +0 0 0	0 0 $\alpha_1\lambda_1$ 0 0 0
0 0	0 0 0* $\mu_1$ 0	0 0 0 $\alpha_2\lambda_2$ $\mu_1$ 0
0 0	0 0 0 $\alpha_2\lambda_1$ * $\mu_2$	0 0 0 $\alpha_2\lambda_1$ $\alpha_2\lambda_2$ $\mu_2$
0 0	0 0 0 $\alpha_2\lambda_2$ 0 *	0 0 0 $\alpha_2\lambda_2$ 0 $\alpha_2\lambda_2$
0 0	0 0 0 0 0 0	* $\mu_1$ 0 0 0 0
0 0	0 0 0 0 0 0	0 * $\mu_2$ 0 0 0
	$\mu_3$ 0 0 0 0	0 0 +0 0 0
	0 0 0 0 0	0 0 0* $\mu_1$ 0
	0 0 0 0 0	0 0 0 $\alpha_2\lambda_1$ * $\mu_2$
	0 0 0 0 0	0 0 0 $\alpha_2\lambda_2$ 0 *
0 0	0 0 0 0 0 0	0 0 0 0 0 0
⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮	⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Where the elements marked by an asterisk in each diagonal block are given by

$$diag\{-(\lambda_1 + \mu_1) \quad -(\lambda_1 + \mu_2) \quad -(\lambda_1 + \mu_3) - (\lambda_2 + \mu_1) \quad -(\lambda_2 + \mu_2) \quad -(\lambda_2 + \mu_3)\}$$

(William, 2009)

**PERFORMANCE MEASURES FOR *ph/ph/1* QUEUES**

Given the stationary probabilities of underlying Markov chain,  $\pi_k$ . The probability that there are  $k$  customers present in the queueing system, for  $k \geq 1$ , is obtained by adding the components of the  $k^{th}$  sub vector, i.e.

$$p_k = \|\pi_k\|_1 = \|\pi_1 R^{k-1}\|_1$$

The probability that the system is empty is given by

$$p_0 = \|\pi_0\|_1.$$

The probability that the system is busy is given by

$$1 - p_0 = 1 - \|\pi_0\|_1 = 1 - \rho.$$

The probability that there  $k$  or more customers present is,

$$\text{Prob } [N \geq k] = \sum_{j=k}^{\infty} \|\pi_j\|_1 = \|\pi_1 \sum_{j=k}^{\infty} R^{j-1}\|_1 = \|\pi_1 R^{k-1} \sum_{j=0}^{\infty} R^j\|_1 = \|\pi_1 R^{k-1} (1 - R)^{-1}\|_1$$

By Little’s law and PASTA property, the average number of customers in phase type  $Ph/Ph/1$  is

$$E[N] = k \sum_{k=1}^{\infty} \|\pi_k\|_1 = \sum_{k=1}^{\infty} k \|\pi_1 R^{k-1}\|_1 = \|\pi_1 \sum_{k=1}^{\infty} \frac{d}{dR} R^k\|_1 = \|\pi_1 \frac{d}{dR} (\sum_{k=1}^{\infty} R^k)\|_1$$

$$E[N] = \|\pi_1 \frac{d}{dR} (1 - R)^{-1}\|_1 = \|\pi_1 (1 - R)^{-2}\|_1$$

Lucantoni, (1993)

From the standard formula,

Mean number of customers waiting in the queue,

$$E[N_q] = E[N] - \frac{\lambda}{\mu}$$

The average response time,  $E[R] = \frac{E[N]}{\lambda} = E[W_q]$

The average time spent waiting in the queue, =

$$E[W_q] = \frac{E[N_q]}{\lambda}$$

**PRACTICAL PROBLEM FORMULATION AND RESULTS**

Given the  $M/E_r/1$  queue with parameters  $\lambda = 1, \mu = 1.5$ , and  $r = 3$ . The infinitesimal generator  $Q$  and its sub-matrices are as follows:

-1	1 0 0	0 0 0	0 0 0	...
0	-5.5 4.5 0	1 0 0	0 0 0	...
0	0 -5.5 4.5	0 1 0	0 0 0	...
4.5	0 0 5.5	0 0 1	0 0 0	...
0	0 0 0	-5.5 4.5 0	1 0 0	...
0	0 0 0	0 -5.5 4.5	0 1 0	...
0	4.5 0 0	0 0 5.5	0 0 1	...
0	0 0 0	0 0 0	-5.5 4.5 0	...
0	0 0 0	0 0 0	0 -5.5 4.5	...
0	0 0 0	4.5 0 0	0 0 5.5	...
⋮	⋮	⋮	⋮	⋮

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4.5 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} -5.5 & 4.5 & 0 \\ 0 & -5.5 & 4.5 \\ 0 & 0 & -5.5 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_{00} = -1, B_{01} = (1 \ 0 \ 0), B_{01} = \begin{pmatrix} 0 \\ 0 \\ 4.5 \end{pmatrix}$$

Our quest is to compute the stationary probability vector  $\pi$  from the system of homogeneous linear equation  $\pi Q = 0$  using the matrix geometric method.

Let the stationary probability vector  $\pi = (\pi_0, \pi_1, \dots, \pi_k, \dots)$ , where  $\pi_0$  is a vector of length 1 (whose value is equal to the probability of an empty system) and  $\pi_k, k = 1, 2, \dots$  is a row vector of length  $r$ . Its  $i$ th component gives the probability of being in phase  $i$  when there are  $k$  customers in the system.

$\therefore \pi_{i+1} = \pi_i R$  for,  $i = 1, 2, 3, \dots$  where  $R$  is the Neut's rate matrix.

By successive substitution (iterative method)

$$R_{L+1} = -(V + R_L^2), \quad L = 0, 1, 2, \dots$$

Where  $V = A_2 A_L^{-1}$  and  $W = A_0 A_L^{-1}$  and using  $R_0 = 0$  to initiate the procedure.

Given that  $\lambda_1 = 1, \lambda_2 = 0.5, \mu = 4, \gamma_1 = 5, \gamma_2 = 3$

The infinitesimal generator is  $Q =$

$\begin{pmatrix} -6 & 5.0 \\ 3 & -3.5 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & & \\ & & 0.5 \end{pmatrix}$		
$\begin{pmatrix} & & \\ 2 & 2.0 & \\ & & \end{pmatrix}$	$\begin{pmatrix} -6 & 5 & \\ 3 & -12 & 5 \\ 0 & 3 & -3.5 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & & \\ & & 0.5 \end{pmatrix}$	
	$\begin{pmatrix} & & \\ & 4 & \\ & & \end{pmatrix}$	$\begin{pmatrix} -6 & 5 & \\ 3 & -12 & 5 \\ 0 & 3 & -3.5 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & & \\ & & 0.5 \end{pmatrix}$
	$\backslash$	$\backslash$	$\backslash$

**Step 1:** To verify the stability of the system

$$A = A_0 + A_1 + A_2.$$

$$A = \begin{pmatrix} & & \\ & 4 & \\ & & \end{pmatrix} + \begin{pmatrix} -6 & 5 & \\ 3 & -12 & 5 \\ 0 & 3 & -3.5 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & & \\ & & 0.5 \end{pmatrix} = \begin{pmatrix} -5 & 5 & 0 \\ 3 & -8 & 5 \\ 0 & 3 & -3 \end{pmatrix}$$

$\therefore$  It has quasi birth – death structure.

Given the stationary probability vector  $\pi_A = (0.1837 \ 0.3061 \ 0.5102)$

and

$$0.4388 = \pi_A A_2 e < \pi_A A_0 e = 1.2245$$

**Step 2:** To get  $R$ ,

$$A^{-1} = \begin{pmatrix} -0.2466 & -0.1598 & -0.2283 \\ -0.0959 & -0.1918 & -0.2740 \\ -0.0822 & -0.1644 & -0.5205 \end{pmatrix}$$

$$V = A_2 A^{-1} = \begin{pmatrix} -0.2466 & -0.1598 & -0.2283 \\ 0 & 0 & 0 \\ -0.041 & -0.0822 & -0.2603 \end{pmatrix}$$

$$W = A_0 A^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -0.3836 & -0.7671 & -1.0959 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_{l+1} = -(V + R_l^2 W), \quad l = 0, 1, 2, \dots$$

By iteration

$$R_{148} = \begin{pmatrix} 0.2917 & 0.2500 & 0.3571 \\ 0 & 0 & 0 \\ 0.625 & 0.125 & 0.3214 \end{pmatrix}$$

**Step 3:** Boundary condition

$$(\pi_0 \ \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + R A_0 \end{pmatrix} = \begin{pmatrix} -6 & 5.0 & : & 1 & 0 & 0 \\ 3 & -3.5 & : & 0 & 0 & 5 \\ 0 & 0 & & -6 & 6.0 & 0 \\ 2 & 2 & : & 3 & -12 & 5.0 \\ 0 & 0 & & 0 & 3.5 & -3.5 \end{pmatrix} = (0 \ 0)$$

By replacing the last equation with the equation  $\pi_{01} = 1$ , i.e., setting the first component of the sub-vector  $\pi_0$  to 1.

The system of equation becomes

$$(\pi_0 \ \pi_1) \begin{pmatrix} -6 & 5.0 & \vdots & 1 & 0 & 1 \\ 3 & -3.5 & \vdots & 0 & 0 & 0 \\ 0 & 0 & & -6 & 6.0 & 0 \\ 2 & 2 & \vdots & 3 & -12 & 0 \\ 0 & 0 & & 0 & 3.5 & 0 \end{pmatrix} \\ = (0 \ 0 \vdots \ 0 \ 0 \ 1)$$

$$\therefore (\pi_0 \ \pi_1) \\ = (1.0 \ 1.6923 \vdots \ 0.3974 \ 0.4615 \ 0.9011)$$

**Step 4:** The normalization constant is  $\alpha = \pi_0 e + \pi_1(1 - R)^{-1}e$

$$= (1.0 \ 1.6923)e \\ + (0.3974 \ 0.4615 \ 0.9011) \begin{pmatrix} 1.4805 & 0.4675 & 0.7792 \\ 0 & 1 & 0 \\ 0.1364 & 0.2273 & 0.1545 \end{pmatrix}$$

$$\therefore \frac{\pi_0}{\alpha} = (0.1678 \ 0.2840),$$

$$\frac{\pi_1}{\alpha} = (0.0667 \ 0.0775 \ 0.1512).$$

$$\pi_2 = \pi_1 R \\ = (0.0667 \ 0.0775 \ 0.1512) \begin{pmatrix} 0.2917 & 0.2500 & 0.3571 \\ 0 & 0 & 0 \\ 0.625 & 0.125 & 0.3214 \end{pmatrix} \\ = (0.0289 \ 0.356 \ 0.0724)$$

$$\pi_3 = \pi_2 R \\ = (0.0289 \ 0.056 \ 0.724) \begin{pmatrix} 0.2917 & 0.25 & 0.3571 \\ 0 & 0 & 0 \\ 0.0625 & 0.125 & 0.3214 \end{pmatrix} \\ = (0.013 \ 0.0356 \ 0.0336) \text{ e. t. c.}$$

Agboola (2011)

**DISCUSSION AND CONCLUSION**

The stationary probability vector  $\pi$  from the system of homogeneous linear equation  $\pi Q = 0$  of a queueing system comprising of two- phase hyper-exponential inter-arrival time, Erlang-3 distributed service time, single server and infinite number of arrivals has been investigated in order to provide some insight into the analysis of more complex phase type distributions PH/PH/1 queues. The result

obtained for this illustrative example is (0.013 0.0356 0.0336), where  $\pi_0 = 0.013$  is a vector of length 1 (whose value is equal to the probability of an empty system) and  $\pi_1 = 0.0356$  and  $\pi_2 = 0.0336$  is a row vector of length  $r$ . Its  $i$ th component gives the probability of being in phase  $i$  when there are 3 customers in the system.

In this model, the infinitesimal generator matrix is obtained through the transition state diagram and probability vector through Neut’s rate matrix R after testing for stability, boundary condition and normalization. Performance Measures for phase type Queueing system, Ph/Ph/1, are obtained as follows, The probability that there are  $k$  customers present in the queueing system, for  $k \geq 1$ , is  $p_k = \|\pi_k\|_1 = \|\pi_1 R^{k-1}\|_1$

The probability that the system is empty is given by  $p_0 = \|\pi_0\|_1$ , The probability that the system is busy is given by  $1 - p_0 = 1 - \|\pi_0\|_1 = 1 - \rho$ , The probability that there  $k$  or more customers present is,  $\text{Prob} [N \geq k] = \sum_{j=k}^{\infty} \|\pi_j\|_1 = \|\pi_1 \sum_{j=k}^{\infty} R^{j-1}\|_1 = \|\pi_1 R^{k-1} \sum_{j=0}^{\infty} R^j\|_1 = \|\pi_1 R^{k-1} (1 - R)^{-1}\|_1$ . By

Little’s law and PASTA property, the average number of customers in phase type  $Ph/Ph/1$  is  $E[N] = k \sum_{k=1}^{\infty} \|\pi_k\|_1 = \sum_{k=1}^{\infty} k \|\pi_1 R^{k-1}\|_1 = \|\pi_1 \sum_{k=1}^{\infty} \frac{d}{dR} R^k\|_1 = \|\pi_1 \frac{d}{dR} (\sum_{k=1}^{\infty} R^k)\|_1$   $E[N] = \|\pi_1 \frac{d}{dR} (1 - R)^{-1}\|_1 = \|\pi_1 (1 - R)^{-2}\|_1$  and from the standard formula, Mean number of customers waiting in the queue,  $E[N_q] = E[N] - \frac{\lambda}{\mu}$ , the average response time,  $E[R] = \frac{E[N]}{\lambda} = E[W_q]$

while The average time spent waiting in the queue,  

$$= E[W_q] = \frac{E[N_q]}{\lambda}$$

The practical examples were then considered for certain given values of  $\lambda$ ,  $\mu$ , and  $\gamma$ . The result obtained for this illustrative example is (0.013 0.0356 0.0336), where  $\pi_0 = 0.013$  is a vector of length 1 (whose value is equal to the probability of an empty system) and  $\pi_1 = 0.0356$  and  $\pi_2 = 0.0336$  is a row vector of length  $r$ .

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