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On the Existence and Stability of Periodic Solutions of Airy's Equation with Elastic Coefficients

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ABSTRACT

In this paper, the existence and stability of periodic solutions of a certain second order differential equation with elastic coefficient were investigated using power series method, eigenvalue approach and Lyapunov direct method. Existence of analytical solution which is independent of time was achieved using the power series method. Eigenvalue approach and Lyapunov direct method were used to investigate the stability of the resulting solution. Periodic solution was obtained using the eigenvalues of the resulting matrix. The first stability method further examined stability of the equilibrium point by considering the intervals around the origin and its discriminant. The equilibrium points for the intervals and the discriminant were unstable because the real part of the characteristic root is zero. Unstable equilibrium point was also obtained for the second stability method using the energy function and time derivative around the equilibrium point. The two unstable results indicated that there were highly instability regions with a strictly positive elastic coefficient. The highly instability regions were confirmed by the presence of elastic coefficient which reduces oscillation with an increase in amplitude. Furthermore, numerical simulations for existence and stability of Airy's equation at different values of the elastic coefficient were illustrated in order to demonstrate the behaviour of the solutions which extends some results in literature.

Keywords: Power series; Lyapunov direct method; Eigenvalue approach; Existence; Stability

Introduction

Consider Airy's equation of the form

$$\ddot{y} + k^2xy = 0 \quad (1.1)$$

where \ddot{y} is the second derivative or acceleration, k^2x is the coefficient of y and k^2 is the elastic coefficient. Aspinen (1966), opined the Airy's equation as the simplest equation with a turning point. The character of the solution changes from oscillatory to exponential and acts like a solution to Schrodinger equation. In Cengel and Palm (2013), the application of Airy's equation is

evident in optics, parabolic equation and astronomy. The search for solution of Airy's type has attracted the attention of many researchers. For instance, see (Olver (1974); Saddiq (2013); Boggarapu (2015)) and their references therein.

Due to the relevance of Airy's equation in real world problem, the study of existence of solutions of equation with variable coefficients has continued to attract the attention of many researchers see (Abramowitz and Stegun (1972); Jung S.M.

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(2008)). On the stability analysis of second order differential equations see (Csoro and Hatrani (2010); Oyesanya and Nwamba (2013); Qiubao (2012), Eze *et al.* (2017); Eze and Obasi, (2017); Eze *et al.* (2019); Eze *et al.* (2019)). Much work has been done on existence of periodic solution of Airy's type using power series method but none have been done on existence and stability of periodic solutions using a combination of power series, eigenvalue approach and lyapunov direct method. This paper is motivated by studying the works of Knaust (1998), Mathias and Ronald (2004), Jung (2008) and Smith (1969). The objectives of this paper therefore are to investigate the existence and stability of periodic solutions of the eqn (1.1)

Materials and Methods

Definition 2.1 (Power series): This is a method used to seek a series solution to certain differential equations. In general, series solution has coefficients which when replaced into the differential equation yields the recurrence relation for the coefficients.

Definition 2.2 (Power series about the point zero): it is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad (1.3)$$

where $a_0, a_1, a_2 \dots a_n$ are real constants

Definition 2.3 (Power series about the point x_0):

This is a series defined as

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \dots \quad (1.4)$$

where $a_0, a_1, a_2 \dots a_n$ are real constants

Definition 2.4 (Convergence of Power series): A series solution is said to converge at a point x if its n th partial sum $\sum_{k=0}^n a_k x^k$ converge that is to say that the limits $L = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k x^k$ exists. The sum of the series is the limit and x is called point of convergence.

Note: $x = 0$ is the point of convergence of equation (1.3)

For example

(i) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(ii) $\sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + x^3 + \dots$

(i) Converges for every value of x in \mathbb{R} and (ii) converges only for $|x| < 1$. The convergence point of (i) and (ii) forms an interval. Moreover, there exists $0 \leq R \leq \infty$ such that the series solution of (i) converges for all $|x - x_0| < R$ while (ii) diverges for all $|x - x_0| > R$. Here R is called radius of convergence. In many cases, R is obtained using the formula $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ whenever the limits exist.

Definition 2.5 (Differential of power series): Suppose eqn (1.3) converges for $|x| < R$ with $R > 0$ and denote the sum by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad (1.5)$$

Then $f(x)$ is automatically continuous and has the derivatives of all order for $|x| < R$. Also

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots \quad (1.6)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2 a_3 x + \dots \quad (1.7)$$

and so on. Each of the resulting series converges for $|x| < R$. The coefficient a_n can be linked to $f(x)$ and its derivative via the following formula

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (1.8)$$

Theorem 2.6 Let $x = 0$ be an equilibrium point of $x = f(x)$ and $V: D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable such that

- i. $V(0) = 0$
- ii. $V(x) > 0 \forall x \in D \setminus \{0\}$ and
- iii. $Lf V(x) \leq 0 \forall x \in D \setminus \{0\}$. Then $x = 0$ is stable. Moreover if $Lf V(x) < 0$ for all $x \in D \setminus \{0\}$ then $x = 0$ is asymptotically stable.

Theorem 2.7 Consider the autonomous system defined by $\frac{dx}{dt} = f(x)$ with $x = 0$ as equilibrium point. Let $J(x)$ denote the Jacobian Matrix of the system such that $J(x) = \frac{\partial f}{\partial x}$

If the matrix $F = J + J^T$ is negative definite, $x = 0$ is asymptotically stable. A energy function for $f(x)$ is $V(x) = f^T(x) f(x)$. If $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then $x = 0$ is globally asymptotically stable.

Definition 2.8 (Eigenvalue Stability Approach)

This concept uses eigenvalue to investigate whether the system of differential equation is stable or not. The stability investigation uses the real part and the imaginary part along with their signs. To investigate the stability of the equilibrium point:

- (i) Linearize the differential equation about the equilibrium point to obtain a matrix
- (ii) Determine the fixed points.
- (iii) Determine the eigenvalue of the fixed point $|A - I\lambda|$ where A is the matrix.
- (iv) Determine the stability using the sign of the eigenvalue.

Definition 2.9 (Stability) The point $x = 0$ is stable if for each $\epsilon > 0$ there exist a $\delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \epsilon$ for all $t \geq 0$

Definition 2.10 (Equilibrium Point) $x_e \in R^n$ is an equilibrium point of a system if $f(x_e) = 0$ where $f: R^n \rightarrow R^n$ or $x(t) = x_e$ is a trajectory.

Theorem 2.11 (Lyapunov function) Let $V(x)$ be a function with the following properties

- 1. $V(\bar{x}) = 0$
- 2. $V(x)$ is positive definite for $x \neq \bar{x}$
- 3. $\dot{V}(x)$ is negative semi-definite along trajectories of $\dot{x} = f(x)$.

Then $V(x)$ is known as a Lyapunov function.

$$\dot{V}(x) = \frac{\partial v}{\partial x} \dot{x} = \sum_i \frac{\partial v}{\partial x_i} f_i(x) \leq 0$$

where $\frac{\partial v}{\partial x} = \left[\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \dots, \frac{\partial v}{\partial x_n} \right]$

Definition 2.12 Lyapunov function is a scalar function used to prove the stability of equilibrium point of an ordinary differential equation. The scalar function says $V(y)$ is defined on a region that is continuous, positive definite ($V(y) > 0 \forall y \neq 0$) and has continuous first partial derivative at every point of domain of the function.

Results

Power Series Solution of Airy’s Equation

We consider eqn (1.1)

$$\ddot{y} + k^2xy = 0 \tag{1.9}$$

Eqn (1.9) has solution

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots = \sum_{n=0}^{\infty} a_nx^n \tag{1.10}$$

Differentiating eqn (1.10) term by term gives

$$\dot{y} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots = \sum_{n=1}^{\infty} na_nx^{n-1} \tag{1.11}$$

$$\dot{y} = 2a_2 + 3.2a_3x + \dots + n(n-1)a_nx^{n-2} + \dots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} \tag{1.12}$$

Substituting for eqn (1.11) and eqn (1.12) into eqn (1.9) gives

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + k^2x(\sum_{n=0}^{\infty} a_nx^n) = 0 \tag{1.13}$$

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + k^2x\sum_{n=0}^{\infty} a_nx^n = 0 \tag{1.14}$$

Evaluating for each value of n for each term in eqn (1.14) we have

$$2a_2 + 3.2a_3x + 4.3a_4x^2 + \dots + a_0k^2x + a_1k^2x^2 + a_2k^2x^3 + a_3k^2x^4 + a_4k^2x^5 + \dots = 0 \tag{1.15}$$

For the power to vanish identically over an interval, each coefficient in the series must be zero.

For x^0 : $2a_2 = 0 \Rightarrow a_2 = 0$

For x^1 : $6a_3 + a_0k^2 = 0 \Rightarrow a_3 = \frac{-a_0k^2}{6}$

For x^2 : $12a_4 + a_1k^2 = 0 \Rightarrow a_4 = \frac{-a_1k^2}{12}$

Hence the solution $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$ gives

$$y = a_0 + a_1t + \frac{(-a_0k^2)}{6}x^3 + \frac{(-a_1k^2)}{12}x^4 + \dots$$

$$= a_0 \left(1 - \frac{k^2x^3}{6} + \dots\right) + a_1 \left(x - \frac{k^2x^4}{12} + \dots\right) \quad (1.16)$$

where a_0 and a_1 are arbitrary constants. Eqn. (1.16) is the solution of eqn (1.1) that is independent of time. The series solution does not terminate as the power of the series solution increases.

Stability Analysis

Eigenvalue Approach

We consider eqn (1.1)

$$\ddot{y} + k^2xy = 0 \quad (1.17)$$

When $k^2 = 1$, eqn (1.17) becomes

$$\ddot{y} + xy = 0 \quad (1.18)$$

Eqn (1.18) is the simplest form of eqn (1.17)

Let $y = y_1, \dot{y}_1 = y_2, \ddot{y}_1 = \dot{y}_2$ then eqn (1.17) becomes

$$\dot{y}_2 + k^2xy_1 = 0 \quad (1.19)$$

Hence the first equivalent system is

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -k^2xy_1 \end{aligned} \quad (1.20)$$

At $\dot{y}_1 = \dot{y}_2 = 0, 0 = (-k^2x)y_1$. The equilibrium point corresponding to eqn (1.20) is (0,0). In matrix form, eqn (1.20) becomes

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2x & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (1.21)$$

where $\dot{x} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -k^2x & 0 \end{bmatrix}$ and $x = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

The stability of the critical point for the linear system is connected with the eigenvalues of A which are given by $|A - I\lambda| = 0$ where λ is the eigenvalue of A .

Evaluating the eigen values we have

$$\left| \begin{bmatrix} 0 & 1 \\ -k^2x & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \quad (1.22)$$

which on further simplification gives

$$\lambda^2 + k^2x = 0 \quad (1.23)$$

The eigenvalues for eqn (1.23) are

$$\lambda_1 = i\sqrt{k^2x} \quad \text{and} \quad \lambda_2 = -i\sqrt{k^2x}$$

Since the real part is zero, the solution to eqn (1.23) is

$$y(t) = A\cos \sqrt{k^2x} t + B\sin \sqrt{k^2x} t \quad (1.24)$$

Eqn (1.24) shows that solution of eqn (1.17) is periodic with the angular velocity given by $\sqrt{k^2x}$. To examine the stability at equilibrium point we consider $\lambda_{1,2} = \pm i\sqrt{k^2x}$ with the following cases:

Case I:

When $x = 0, \lambda_{1,2} = 0$ corresponds to instability with no oscillatory behavior

Case II

When $x > 0, \lambda_{1,2} = \pm ik$ corresponds to critical points that are centers for which stability is ensured.

Case III

When $x < 0, \lambda_{1,2} = \pm i^2k$ corresponds to saddles giving rise to instability.

For the discriminate:

Case I:

When $b^2 - 4ac < 0$, the result follows from eqn (1.23) where the damping coefficient is zero. The system will undergo an oscillation but with decreasing amplitude which come to rest at equilibrium.

Case II:

When $b^2 - 4ac = 0$, the system come to equilibrium as fast as possible.

When $b^2 - 4ac > 0$, the system does not come to rest as fast as possible due to increasing amplitude.

From the above cases, the equilibrium point (0,0) is unstable because of the zero-real part. The presence of the elastic coefficient in (1.24) reduces oscillation with decreasing amplitude. Hence, elastic coefficient has a damping effect.

Lyapunov Approach

Multiplying eqn. (1.18) by \dot{y} we have

$$\dot{y}\ddot{y} + k^2xy\dot{y} = 0 \tag{1.25}$$

But $\frac{d}{dt}(\dot{y})^2 = 2\dot{y}\ddot{y} \implies \dot{y}\ddot{y} = \frac{1}{2}\frac{d}{dt}(\dot{y})^2$.

Substituting for $\dot{y}\ddot{y}$ in (1.25) we have

$$\frac{1}{2}\frac{d}{dt}(\dot{y})^2 + k^2xy\dot{y} = 0 \tag{1.26}$$

Integrating (1.26) we have

$$\frac{1}{2}(\dot{y})^2 + k^2xy^2 = C \tag{1.27}$$

Since total energy in a system is conserved then $\frac{dH}{dt} = 0$. By integration $H = C$ where H is the energy function. Therefore $H(y, \dot{y}) = \frac{1}{2}(\dot{y})^2 + k^2xy^2$. Hence the Lyapunov function is

$$V(y_1, y_2) = \frac{k^4x^2y_1^2}{2} + k^2xy_1^2 \tag{1.28}$$

For stability analysis by Lyapunov approach we have

$$V(0,0) = 0,$$

$$V(y_1, y_2) > 0$$

$$\dot{V}(y_1, y_2) = \frac{\partial V}{\partial y_1} \frac{dy_1}{dt} + \frac{\partial V}{\partial y_2} \frac{dy_2}{dt} = k^4x^2y_1y_2 > 0.$$

Hence (0,0) is unstable.

Numerical Simulations for Existence of solution of Airy's Equation

$$k := 0.1$$

Define a function that determines a vector of derivative values at any solution point (t,Y):

$$D(t,X) := \begin{bmatrix} X_1 \\ -(1-t) \cdot X_0 \end{bmatrix}$$

Define additional arguments for the ODE solver:

$$t0 := 0$$

Initial value of independent variable

$$t1 := 50$$

Final value of independent variable

$$x0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Vector of initial function values

Number of solution values on [t0, t1]

Solution matrix:

$$S := \text{Rkadapt}(X0, t0, t1, N, D)$$

$$t := S^{(0)}$$

Independent variable values

$$x1 := S^{(1)}$$

First solution function values

$$x2 := S^{(2)}$$

Second solution function values

Table 1: Table of values for the independent variables

	0	1	2
0	0	0	1
1	0.033	0.033	0.999
2	0.067	0.067	0.998
3	0.1	0.1	0.995
4	0.133	0.133	0.992
5	0.167	0.166	0.988
6	0.2	0.199	0.983
7	0.233	0.231	0.977
8	0.267	0.264	0.971
9	0.3	0.296	0.964
10	0.333	0.328	0.957
11	0.367	0.36	0.95
12	0.4	0.392	0.942
13	0.433	0.423	0.934
14	0.467	0.454	0.926
15	0.5	0.485	...

S =

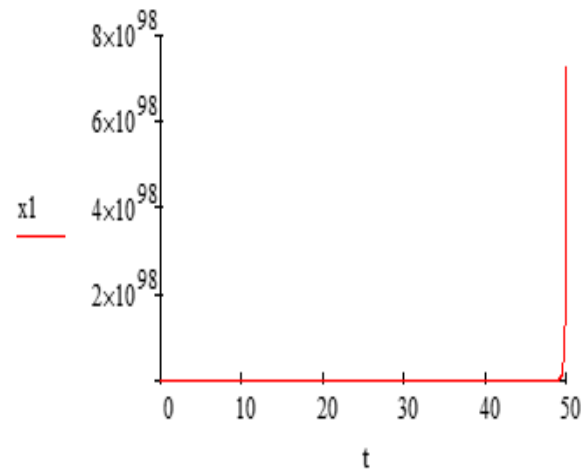


Figure 1: The graph of x1 against t showing increase in the series solution as powers in the series increases. The trajectory parallel to x1 axis shows that the series does not vary with respect to time.

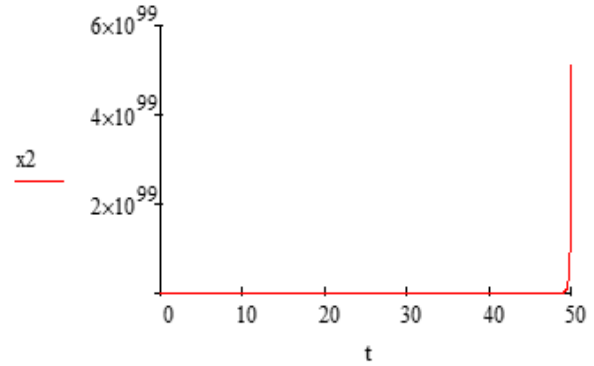


Figure 2: The graph of x2 against t showing increase in the series as the power of x2 values increases. The trajectory which is parallel to the x2 axis showed that the series solution increases faster due to increase in the power of the series.

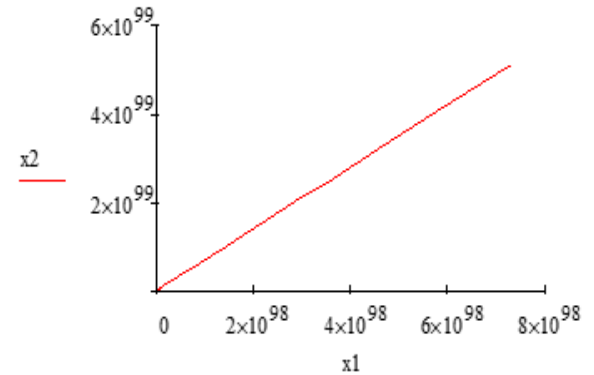


Figure 3: The graph of x2 and x1 showing variations on the axis. This shows existence of power series solution which is purely independent of time. The trajectory increases as the power of the axis increases.

Numerical Simulations of Airy’s Equation

(i) $k := 0.5$

Define a function that determines a vector of derivative values at any solution point (t,Y):

$$D(x, Y) := \begin{pmatrix} Y_1 \\ -k^2 \cdot x \cdot Y_0 \end{pmatrix}$$

Define additional arguments for the ODE solver:

$$x0 := 0$$

Initial value of independent variable

$$x1 := 50$$

Final value of independent variable

$$Y0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Vector of initial function values

$$N := 1500$$

Number of solution values on [x0, x1]

Solution matrix:

$$S := \text{Rkadapt}(Y0, x0, x1, N, D)$$

$$x := S^{\langle 0 \rangle}$$

Independent variable values

$$y1 := S^{\langle 1 \rangle}$$

First solution function values

$$y2 := S^{\langle 2 \rangle}$$

Second solution function values

Table 2: Solution Matrix for independent variable values.

	0	1	2
0	0	0	1
1	0.033	0.033	1
2	0.067	0.067	1
3	0.1	0.1	1
4	0.133	0.133	1
5	0.167	0.167	1
6	0.2	0.2	0.999
7	0.233	0.233	0.999
8	0.267	0.267	0.998
9	0.3	0.3	0.998
10	0.333	0.333	0.997
11	0.367	0.366	0.996
12	0.4	0.399	0.995
13	0.433	0.433	0.993
14	0.467	0.466	0.992
15	0.5	0.499	...

S =

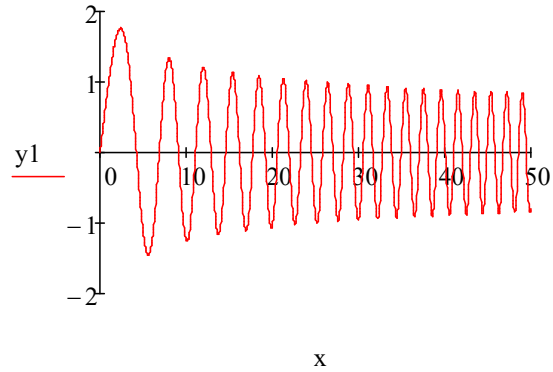


Figure 4: Graphical profile of trajectory $y(x)$ against time showing periodic motion

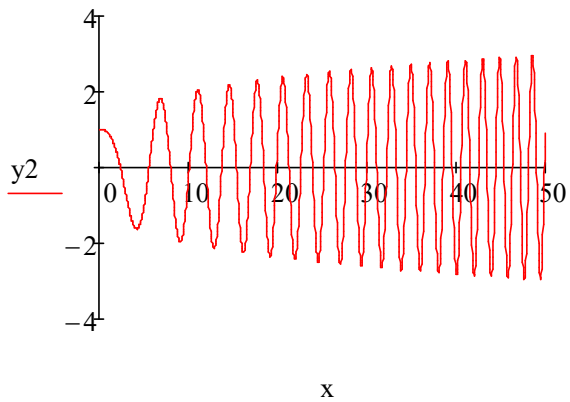


Figure 5: Graphical profile of gradient $y'(x)$ against time showing increase in the periodic motion as the trajectory moves a little distance from the origin.

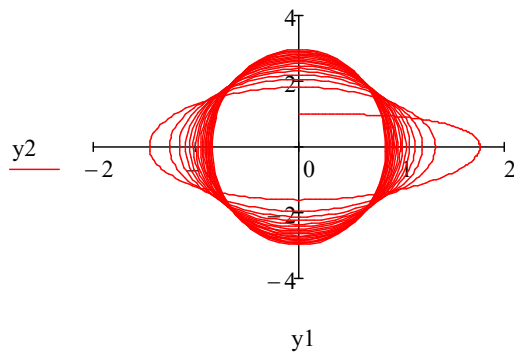


Figure 6: Phase portrait of Airy's equation showing instability.

(ii) $k := 0.08$

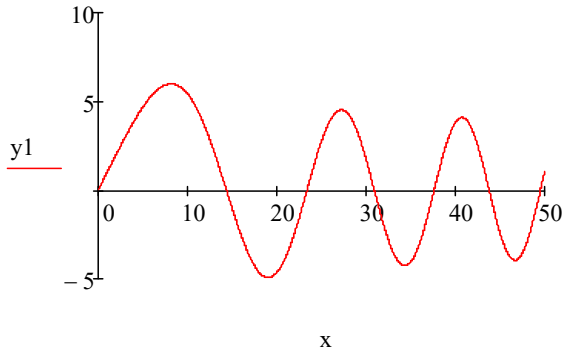


Figure 7: Graphical profile of trajectory $y(x)$ against time showing sinusoidal motion from the origin.

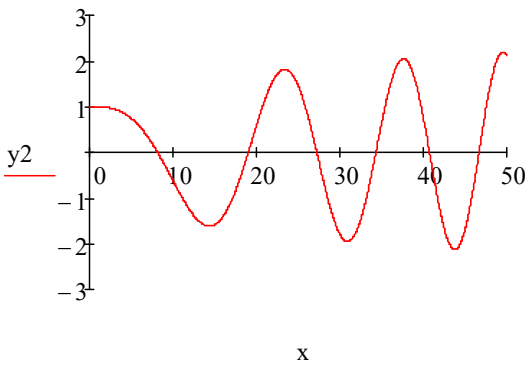


Figure 8: Graphical profile of gradient $y'(x)$ against time showing periodic motion with an increase in amplitude.

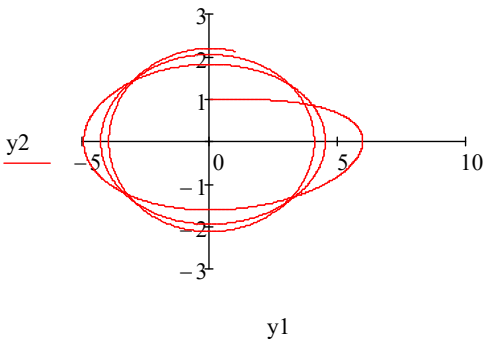


Figure 9: Phase curve of Airy's equation showing instability.

Discussion

Figure 1 describes trajectory without an oscillation. At the point $t = 50$, the trajectory is parallel to the axis of the first solution function values. This behaviour describes the analytic solution of Airy's equation which does not vary for all values of t .

Figure 2 describes the existence of solution of Airy's equation that is not periodic in nature. The increase x_2 axis leads to a constant increase with respect to time which terminate at $t = 50$. Since the trajectory is parallel to the axis of the first solution function values, the elastic coefficient has no effect on the series solution.

In figure 3, the trajectory starts from the origin. The increase in x_1 axis leads to a corresponding increase in x_2 axis. This behavior describes the series solution of Airy's differential equation which is independent of time.

Figure 4 describes a trajectory that is periodic in nature. The periodic motion decrease as it moves away from the equilibrium point with a decrease in amplitude. This behavior describes the unstable nature of Airy's equation with unequal amplitude.

Figure 5 describes a trajectory that is periodic in nature. The trajectory increases as it goes away from the equilibrium point with increase in amplitude. This shows that equilibrium point of the differential equation is unstable.

Figure 6 describes a phase portrait that is conserved with the phase curve away from the equilibrium point. The phase portrait describes a nested curve at different values on the interval. The behavior of phase portrait shows unstable nature of the differential equation since the trajectory is far away from the equilibrium point.

Figure 7 describes sinusoidal motion when $k = 0.08$. The trajectory starts rom the origin and spread out evenly with high amplitude. This shows that decrease in elastic coefficient leads to increase

in oscillation. The trajectory is periodic with an unstable equilibrium point.

Figure 8 describes trajectory which is periodic in nature. This behavior shows that equilibrium point is unstable with unequal amplitude. This invariably explained the high unstable nature of Airy's equation whose time derivative at the origin is greater than zero.

Figure 9 describe a phase portrait with few nested curves. This is due to decrease in the value of k . This behavior shows that the equilibrium point is unstable as the trajectory is far away from the equilibrium point. This type of phase portrait can be seen in parabolic equation where Airy's equation is applied.

Conclusions

From our results, existence and stability of Airy's equation was achieved. Using the power series method, the real valued functions which are analytic shows that the solution exists. This is because Airy's equation has variable coefficient along with the dependent variable. With the help of characteristic equation, periodic solution was achieved. Stability analysis by Eigenvalue approach and Lyapunov method was applied to the differential equation. Precise results for the dynamical system was obtained using analytical calculations. Eigenvalue approach for investigating stability holds good for linear systems because linear systems have only one equilibrium point. Similar results obtained for stability investigation shows that the equilibrium point of Airy's equation is highly unstable. The unstable nature was found in figures 6 and 9 where the phase curves are far from the equilibrium point. The elastic coefficient which is strictly positive shows that the damping coefficient is high with decreasing amplitude. Our conclusion is that Airy's equation has different areas of applications.

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Competing Interest

No conflict of Interest exists.

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