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On the Application of Mean Value Analysis Without the Normalization Constant in the Closed Queuing Networks

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ABSTRACT

An example of closed queue network could be view when patients arrive at a doctor's office to update their medical records, then it's off to the nurse's station for various measurements like weight, blood pressure, and so on. The next step is generally to queue (i.e., wait patiently) for one of the doctors to arrive and begin the consultation and examination. Perhaps it may be necessary to have some X-rays taken, an ultrasound may be called for, and so on. After these procedures have been completed, it may be necessary to talk with the doctor once again. The final center through which the patient must pass is always the billing office. In this work, multiple-node" system in which a customer requires service at more than one node, which may be viewed as a network of nodes, and each node is a service center having storage room for queues to form and perhaps with multiple servers to handle customer requests is investigated in order to provide some insight into the performance measure analysis. Our quest is to exempt the normalization constant in the computation of performance measure in the closed queueing network. The arrival properties and Little's law are use with the help of some existing equations and formulas in queueing network. Performance measures, such as Mean number of customers, response time, throughput, and marginal probability distribution are obtained for central server and load dependent server closed queueing networks for nodes 4 and 5, and also for $k = 3$ and $k = 10$.

Keywords: Convolution Algorithm, Load dependent, Normalization Constant, Throughput and Visit Ratio.

Introduction

The convolution algorithm provides a means of computing the complete set of marginal probabilities of each node in a product-form queueing network. The disadvantage of this method is that it requires forming the normalization constant and this can be numerically unstable, as well as time consuming. Furthermore, in some circumstances, the complete set of marginal probabilities is not needed; all that is required in many cases are mean values, such as the

mean number of customers at each node. The mean value analysis (MVA) approach allows us to compute mean values without having to first compute a normalization constant. The study of closed queueing system with exponential servers was first introduced by Gordon and Newell (1967) and this was extended to computational algorithms for closed queueing networks with exponential servers by Buzen (1973), Chang and Lavenberg (1974) introduced the work rate in closed queueing

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networks with general independent servers, while the Baskett *et al* (1975) discussed open, closed and mixed networks with different classes of customers. The application of queue model to computer system with general service time distribution was shown by Schumi (1976). Knuth (1998) studied the art of computer programming in queuing network while Haverkort (1998) analysed the performance of computer communication systems. Law and Kelton (2000) presented the simulation modelling analysis of solving a queue networks while Rose (2003) extended the simulation to solved queue networks and this was investigated and built upon by Stewart (2009) into Probability, Markov Chain, Queues and queuing networks, while Agboola (2016) extended the application of queuing network to solved machine repair problems with multiple servers. Quan-lin (2017) discussed new invention in queuing models, application and network, while Ronald (2018) applied queuing network model developed to the analysis of manufacturing systems and Barcel (2018) derived an exact solution for the steady state system size probabilities.

Notation

The mean value analysis (MVA) of closed queuing network is developed based on the following notation

p_{ij}	The fraction of departures from node i that go next to node j
λ_i	The overall arrival rate to node i
$G(N)$	Normalization constant
$S(N, M)$	The set of all possible states
V_i	Visit ratio at node i
$X_i(k)$	The throughput of node i in a closed network with k customers
$g_m(n)$	Auxiliary function
$L_i(k) = E[N_i(k)]$	Mean number of customers
$R_i(k)$	Response rate

2.0 Materials and Methods

Arrival theorem states that, in a closed queuing network, the stationary distribution observed by a customer arriving at a node of the network (having left one node and about to enter another) is equal to the stationary distribution of customers in the same network with one fewer customer. In other words, at the moment of arrival to a node, a customer in a closed queuing network containing k customers sees the same stationary distribution of customers over the nodes of the network that an external observer sees when the network contains only $k - 1$ customers. This theorem hints at a recursion based on the number of customers in the network. Beginning with an analysis of the network with a single customer, we shall seek to successively add customers one at a time until we have the results for the network with the required number of customers, N . The arrival theorem by itself is insufficient to allow us to do this. We also need to use Little's law, applied to the overall network as well as to the individual nodes of the network. Thus, the mean value analysis (MVA) approach is based on

1. the arrival theorem—to compute the average response time at each node;
2. Little's law applied to the entire network—to compute the overall throughput of the network and from it, the throughput of the individual nodes;
3. Little's law applied to each node—to compute the mean number of customers at each node.

Considering the arrival theorem at the instant a customer, departing from some node in the network, arrives at node i . This customer observes a certain number of other customers already present. The arrival theorem tells us how many. The mean number already present at the moment of arrival is equal to the mean number at this node when only $k - 1$ customers circulate in the network. This mean number is given by $L_i(k - 1)$. Thus the arriving customer finds $L_i(k - 1)$ customers

already present, all of which will be served before the arriving customer, a result of the FCFS scheduling policy. The arriving customer will wait in the queue until all are served, (i.e., $\frac{L_i(k-1)}{\mu_i}$) and will then spend a time period equal to $\frac{1}{\mu_i}$ being served itself. The average response time (per visit to node i) for the arriving customer is equal to the sum of these, i.e.,

$$R_i(k) = \frac{1}{\mu_i}(L_i(k-1) + 1), i = 1, 2, \dots M \quad (1)$$

This result which was derived for exponential servers and first come first served (FCFS) scheduling, is also applicable to processor sharing (PS) nodes and to last come first serve pre-empt resume (LCFS-PR) nodes and in these cases, the service time distributions need not be exponential. For IS nodes, no customer waits for service and in this case the average response time is just equal to the average service time:

$$R_i(k) = \frac{1}{\mu_i}, \quad i = 1, 2, \dots M$$

We now apply Little's law to the entire network. The mean number of customers in the network is simply the fixed number of customers that circulate in the network, i.e., k . By Little's law this must be equal to the product of the arrival rate and the mean response time. Since the network is in steady state, the arrival rate is equal to the overall throughput of the network i.e., the throughput of node 1 which is $X_1(k) = X(k)$. The mean response time is equal to $\sum_{i=1}^M V_i X_i(k)$. Expressing the throughput in terms of the two other quantities, we have

$$X(k) = \frac{k}{\sum_{i=1}^M V_i X_i(k)} \quad (2)$$

Observe that the throughput of the individual nodes may be computed from this overall throughput. We have

$$X_i(k) = V_i X(k), \quad i = 1, 2, \dots M$$

Finally, we need to apply Little's law to the individual nodes of the network. Equating the

arrival rate to the throughput, $X_i(k)$, at node i , we have

$$L_i(k) = X_i(k)R_i(k) = V_i X(k)R_i(k), i = 1, 2, \dots M \quad (3)$$

With these new values of $L_i(k)$, we may return to equation (1) and begin all over again, with k incremented by one. Thus, the MVA algorithm is obtained by recursively applying Equations (1), (2), and (3) beginning with the boundary conditions

$$L_i(0) = 0$$

In this manner we may compute the mean response time, the mean number of customers, as well as the throughput and utilization at each node of the network. What we cannot compute are the marginal distributions at each node. For this we need the normalization constant $G(N)$. However, using the results from the convolution algorithm, throughput could be written as

$$X(k) = \frac{G(k-1)}{G(k)}, \text{ i.e., } G(k) = \frac{G(k-1)}{X(k)}. \quad (4)$$

This may be incorporated into the second step of the MVA algorithm and the normalization constant computed recursively using the initial condition $G(0) = 1$. Once the normalization constant has been obtained, the marginals follow directly from Equation

$$P_i(n, N) = \frac{Y_i^n}{G(n)} G(N - n) - Y_i G(N - 1 - n) \quad (5)$$

We illustrate these concepts by means of the following example.

3.0 Results

Consider the closed queueing network of the central server type illustrated in Figure 1. Node 1 (the central server) is an infinite server with service rate $\mu_1 = 2.0$. Customers leaving the central server go to node 2, 3, or 4 with probability 0.2, 0.3, and 0.5, respectively. These nodes have a single exponential server which provide service at rates $\mu_2 = 1.0, \mu_3 =$

2.0, and $\mu_4 = 4.0$ respectively. Customers leaving nodes 2, 3, and 4 return to the central server.

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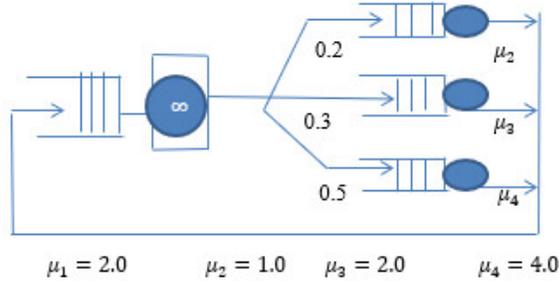


Figure 1: Central server queueing model

We begin the recursive process using Equation (1) with $k = 1$ and obtain

$$R_1(1) = \mu_1^{-1} = 0.5$$

$$R_2(1) = \mu_2^{-1}(L_2(0) + 1) = \mu_2^{-1}(0 + 1) = 1$$

$$R_3(1) = \mu_3^{-1}(L_3(0) + 1) = \mu_3^{-1}(0 + 1) = 0.5$$

$$R_4(1) = \mu_4^{-1}(L_4(0) + 1) = \mu_4^{-1}(0 + 1) = 0.25$$

We next use Equation (2) to compute the overall throughput. We have

$$\begin{aligned} X(1) &= \frac{1}{\sum_{i=1}^4 V_i R_i(1)} \\ &= \frac{1}{(1 \times 0.5) + (0.2 \times 1) + (0.3 \times 0.5) + (0.5 \times 0.25)} \\ &= \frac{1}{0.975} = 1.0256 \end{aligned}$$

At this point we may now compute $G(1)$. We have

$$G(1) = \frac{G(0)}{X(1)} = \frac{1}{1.0256} = 0.975$$

The final calculation for this first round is to use Equation (3) to determine the mean customer population at each node. We have

$$L_1(1) = X(1)V_1R_1(1) = 1.0256 \times 1.0 \times 0.5 = 0.5128$$

$$L_2(1) = X(1)V_2R_2(1) = 1.0256 \times 0.2 \times 1.0 = 0.2051$$

$$L_3(1) = X(1)V_3R_3(1) = 1.0256 \times 0.3 \times 0.5 = 0.1539$$

$$L_4(1) = X(1)V_4R_4(1) = 1.0256 \times 0.5 \times 0.25 = 0.1282$$

We are now ready to begin the second round. Returning to Equation (1), this time using $k = 2$, we obtain

$$R_1(2) = \mu_1^{-1} = 0.5$$

$$R_2(2) = \mu_2^{-1}(L_2(1) + 1) = \mu_2^{-1}(0.2051 + 1) = 1.2051$$

$$R_3(2) = \mu_3^{-1}(L_3(1) + 1) = \mu_3^{-1}(0.1539 + 1) = 0.577$$

$$R_4(2) = \mu_4^{-1}(L_4(1) + 1) = \mu_4^{-1}(0.1282 + 1) = 0.2821$$

Again, we use Equation (2) to compute the overall throughput. We have

$$\begin{aligned} X(2) &= \frac{2}{\sum_{i=1}^4 V_i R_i(2)} \\ &= \frac{2}{(1 \times 0.5) + (0.2 \times 1.2051) + (0.3 \times 0.577) + (0.5 \times 0.2821)} \\ &= \frac{2}{1.975} = 1.8954. \end{aligned}$$

At this point we may now compute $G(2)$. We have

$$G(2) = \frac{G(1)}{X(2)} = \frac{0.975}{1.8954} = 0.5144$$

The final calculation for this second round is to use Equation (3) to determine the mean population size at each node. We have

$$L_1(2) = X(2)V_1R_1(2) = 1.8954 \times 1.0 \times 0.5 = 0.9477$$

$$L_2(2) = X(2)V_2R_2(2) = 1.8954 \times 0.2 \times 1.2051 = 0.4568$$

$$L_3(2) = X(2)V_3R_3(2) = 1.8954 \times 0.3 \times 0.577 = 0.3281$$

$$L_4(2) = X(2)V_4R_4(2) = 1.8954 \times 0.5 \times 0.25 = 0.2674$$

The process continues in this fashion until the network contains the desired number of customers.

Finally, to compute the marginal distribution of customers at node 2, for example, we use the formula for a load-independent node namely,

$$p_i(n, N) = \frac{Y_i^N}{G(N)} [G(N - n) - Y_i G(N - 1 - n)]$$

With $Y_2 = \frac{0.2}{1}$, $G(0) = 1$, $G(1) = 0.975$, $G(2) = 0.5144$, and $G(3) = 0.1970$

$$p_2(0,3) = \frac{(0.2)^0}{G(3)} [G(3) - 0.2G(2)] = 0.4778$$

$$p_2(1,3) = \frac{(0.2)^0}{G(3)} [G(2) - 0.2G(1)] = 0.3243$$

$$p_2(2,3) = \frac{(0.2)^0}{G(3)} [G(1) - 0.2G(0)] = 0.1574$$

$$p_2(3,3) = \frac{(0.2)^0}{G(3)} [G(0)] = 0.406$$

Mean Value Analysis (MVA) and Load-Dependent Service Centers

This is the case that arises when a node contains $c > 1$ identical servers,

Let $\mu_i \alpha_i(k)$ be the service rate when k customers are present. A suitable choice of $\alpha_i(k)$ allows us to handle all possible cases. For example, setting $\alpha_i(k) = \min(c_i, k)$, is what is needed for a node with c_i identical servers. Setting $\alpha_i(k) = \frac{\beta_i(k)}{\mu_i}$ sets the service rate to $\beta_i(k)$. In the load-independent case, it was sufficient to know the mean number of customers present at a node to compute the mean response time, since the server works at the same rate no matter how many are present. In a load dependent node, this is no longer the case: the rate of service is a function of the number of customers present. Now it is necessary to know the probability distribution of customers. We shall let $p_i(j, k)$ denote the equilibrium probability of having j customer present at node i when a total of k customers circulates in the network. These are the marginal probabilities of node i in a network of k customers. Consider what happens when a customer arrives at load-dependent server i and finds $j - 1$, $j = 1, 2, \dots, k$, customers already present. From the arrival theorem, we know that the probability of this happening is given by $p_i(j - 1, k - 1)$. Once the

customer has arrived, the node will contain j customers all of which must be served. With j customers, the mean service time is $\frac{1}{\mu_i \alpha_i(j)}$. It follows then that the response time at node i is given by

$$R_i(K) = \sum_{j=1}^K \frac{1}{\mu_i \alpha_i(j)} p_i(j - 1, k - 1)$$

This equation replaces Equation (1) when the node is load dependent. Of course, this entails a knowledge of the marginal distribution of customers at a load-dependent node and it is to this that we now turn our attention. It is fortunate that these too may be computed from the marginal distribution of customers in a network with one fewer customer. From the results obtained using the convolution algorithm, Equation (6), we saw that the marginal distribution of customers at node i is given by

$$P_i(j, K) = f_i(j) \frac{g_M^i(K-j)}{G(K)} \tag{6}$$

since

$$f_i(j) = \frac{V_i}{\mu_i \alpha_i(j)} f_i(j - 1)$$

And writing $g_M^i(K - j) = g_M^i(K - 1 - j + 1) = g_M^i[(K - 1) - (j - 1)]$

$$P_i(j, K) = \frac{V_i}{\mu_i \alpha_i(j)} f_i(j - 1) \frac{g_M^i[(K - 1) - (j - 1)]}{G(K)}$$

$$P_i(j, K) = \frac{V_i}{\mu_i \alpha_i(j)} * \frac{G(K - 1)}{G(K)} * f_i(j - 1) \frac{g_M^i[(K - 1) - (j - 1)]}{G(K - 1)}$$

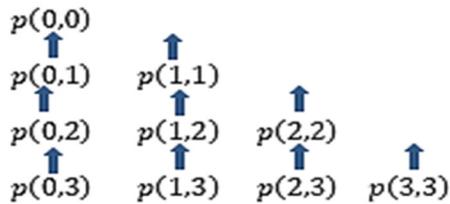
$$P_i(j, K) = \frac{X_i(K)}{\mu_i \alpha_i(j)} * P_i(j - 1, K - 1) \tag{4}$$

Thus, given the throughput of a load-dependent node and its marginal probability distribution when the network has $k-1$ customers, we can compute the marginals when the network has k customers. The algorithm is bootstrapped from the initial condition $p_i(0,0) = 1$, which simply says that there cannot be any customers at service center i if there are none

anywhere in the network. For any value $k > 0$, Equation (4) allows us to compute the probabilities $p_i(j, K)$ for values of $j = 1, 2, \dots, k$. This still leaves the problem of computing $p_i(0, K)$. However, since the sum of the marginal probabilities at any node must be 1, i.e., $\sum_{j=0}^K p_i(j, K) = 1$, we must have

$$p_i(0, K) = 1 - \sum_{j=1}^K p_i(j, K)$$

Unfortunately, this subtraction can lead to a loss of significance and is a major difficulty in implementing a robust version of the MVA algorithm. Observe that it is not necessary to use a full two-dimensional array for the computation of these probabilities. A single vector of length $N + 1$ is sufficient, provided the computation is conducted from the last value to the first. This is illustrated below, where the double up-arrows indicate the positions into which the new probability will be placed. Notice that if the insertions are made from right to left, i.e., $p_i(j, K)$ is computed and inserted before $p_i(j - 1, K)$, then the values needed in the computation of any $p_i(j, K)$ are not overwritten until they are no longer needed.



We now present an example to illustrate these concepts.

Load Dependent Server Illustrative Example

We shall modify the previous example to include a load dependent server. The modified queueing network is shown in Figure 2.

In this example, rather than returning to service center 1, customers departing from service centers 2, 3, and 4 proceed on to the new node. This node, denoted service center 5, contains $m_5 = 2$,

identical exponential servers with parameter $\mu_5 = 2.5$. Thus the load dependent service rate is given by $\mu_j = \mu_5 \alpha_5$ where $\alpha_5(1) = 1$ and $\alpha_5(j) = 2$ for $j > 1$.

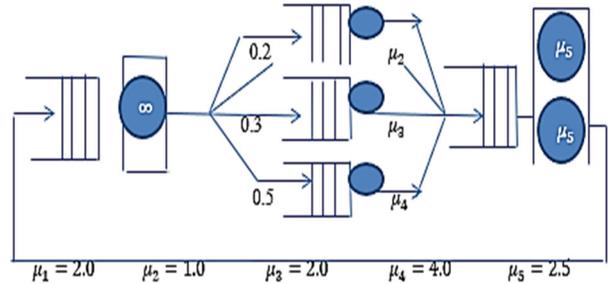


Figure 2: The modified Load Dependent Server Queueing Network of Figure 1.

Customers leaving this service center return to service center 1. As previously, service center 1 is an infinite server with service rate $\mu_1 = 2.0$. Customers leaving this node go to nodes 2, 3, or 4 with probabilities 0.2, 0.3, and 0.5 respectively. These nodes have a single exponential server which provide service at rates $\mu_2 = 1.0$, $\mu_3 = 2.0$, and $\mu_4 = 4.0$ respectively. We begin the recursive process with $k = 1$. From Equation (1), we have

We begin the recursive process using Equation (1) with $k = 1$ and obtain

$$R_1(1) = \mu_1^{-1} = 0.5$$

$$R_2(1) = \mu_2^{-1}(L_2(0) + 1) = \mu_2^{-1}(0 + 1) = 1$$

$$R_3(1) = \mu_3^{-1}(L_3(0) + 1) = \mu_3^{-1}(0 + 1) = 0.5$$

$$R_4(1) = \mu_4^{-1}(L_4(0) + 1) = \mu_4^{-1}(0 + 1) = 0.25$$

$$R_5(1) = \alpha_5(1)\mu_5^{-1}p_5(0,0) = 0.4$$

where we have used the initial condition $p_5(0,0) = 1$. We next use Equation (2) to compute the overall throughput. We have

$$X(1) = \frac{1}{\sum_{i=1}^5 V_i R_i(1)} = \frac{1}{(1 \times 0.5) + (0.2 \times 0.1) + (0.3 \times 0.5) + (0.5 \times 0.25) + (1.0 \times 0.4)} = 0.7273$$

At this point we may now compute $G(1)$. We have

$$G(1) = \frac{G(0)}{X(1)} = \frac{1}{0.7273} = 1.375$$

The final calculation for this first round is to use Equation (3) to determine the mean customer population at each node. We have

$$L_1(1) = X(1)V_1R_1(1) = 0.7273 \times 1.0 \times 0.5 = 0.3636$$

$$L_2(1) = X(1)V_2R_2(1) = 0.7273 \times 0.2 \times 1.0 = 0.1455$$

$$L_3(1) = X(1)V_3R_3(1) = 0.7273 \times 0.3 \times 0.5 = 0.1091$$

$$L_4(1) = X(1)V_4R_4(1) = 0.7273 \times 0.5 \times 0.25 = 0.0909$$

$$L_5(1) = X(1)V_5R_5(1) = 0.7273 \times 1.0 \times 0.4 = 0.2909$$

Before finishing with $k = 1$, we must first compute the marginal for node 5. In particular we need $p_5(0,1)$ and $p_5(1,1)$ We first compute $p_5(1,1)$ from Equation (4). We have

$$p_5(1,1) = \frac{X_5(1)}{\alpha_5(1)\mu_5(1)} p_5(0,0) = \frac{0.7273}{2.5} = 0.2909$$

and, from $p_5(0,1) = 1 - p_5(1,1)$, we have $p_5(0,1) = 0.7091$. We are now ready to begin the second round, this time with $k = 2$. We first compute the mean response time for all five nodes:

$$R_1(2) = \mu_1^{-1} = 0.5$$

$$R_2(2) = \mu_2^{-1}(L_2(1) + 1) = \mu_2^{-1}(0.1455 + 1) = 1.1455$$

$$R_3(2) = \mu_3^{-1}(L_3(1) + 1) = \mu_3^{-1}(0.1091 + 1) = 0.5546$$

$$R_4(2) = \mu_4^{-1}(L_4(1) + 1) = \mu_4^{-1}(0.0909 + 1) = 0.2727$$

$$\begin{aligned} R_5(2) &= \frac{1}{\alpha_5(1)\mu_5} p_5(0,1) + \frac{2}{\alpha_5(2)\mu_5} p_5(1,1) \\ &= \left(\frac{1}{1 \times 2.5} \times 0.7091 \right) \\ &\quad + \left(\frac{1}{2 \times 2.5} \times 0.2909 \right) = 0.4 \end{aligned}$$

Again, we use Equation (2) to compute the overall throughput. We have

$$\begin{aligned} X(2) &= \frac{2}{\sum_{i=1}^4 V_i R_i(1)} \\ &= \frac{2}{(1 \times 0.5) + (0.2 \times 1.1455) + (0.3 \times 0.5546) + (0.5 \times 0.2727) + (1 \times 0.4)} \\ &= 1.3968. \end{aligned}$$

At this point we may now compute $G(2)$. We have

$$G(2) = \frac{G(1)}{X(2)} = \frac{1.375}{1.3968} = 0.9844$$

The final calculation for this second round is to use Equation (3) to determine the mean population size at each node. We have

$$L_1(2) = X(2)V_1R_1(2) = 1.3968 \times 1.0 \times 0.5 = 0.6984$$

$$L_2(2) = X(2)V_2R_2(2) = 1.3968 \times 0.2 \times 1.1455 = 0.3200$$

$$L_3(2) = X(2)V_3R_3(2) = 1.3968 \times 0.3 \times 0.577 = 0.2324$$

$$L_4(2) = X(2)V_4R_4(2) = 1.3968 \times 0.5 \times 0.2727 = 0.1905$$

$$L_5(2) = X(2)V_5R_5(2) = 1.3968 \times 1.0 \times 0.4000 = 0.5587$$

The marginals at node 5 may now be computed. We have, from Equation (4),

$$p_5(1,2) = \frac{X_5(2)}{\alpha_5(1)\mu_5} p_5(0,1) = \frac{1.3968}{1 \times 2.5} \times 0.7091 = 0.3962$$

$$p_5(2,2) = \frac{X_5(2)}{\alpha_5(2)\mu_5} p_5(1,1) = \frac{1.3968}{2 \times 2.5} \times 0.2909 = 0.0813$$

and hence $p_5(0,2) = 1 - p_5(1,2) - p_5(2,2) = 0.5225$. We shall do one more step, now augmenting k to 3. Again, the first quantities to compute, from Equation (1), are the mean response times. We obtain

$$R_1(3) = \mu_1^{-1} = 0.5$$

$$R_2(3) = \mu_2^{-1}(L_2(2) + 1) = \mu_2^{-1}(0.3200 + 1) = 1.3200$$

$$R_3(3) = \mu_3^{-1}(L_3(2) + 1) = \mu_3^{-1}(0.2324 + 1) = 0.6162$$

$$R_4(3) = \mu_4^{-1}(L_4(2) + 1) = \mu_4^{-1}(0.1905 + 1) = 0.2976$$

$$\begin{aligned} R_5(3) &= \frac{1}{\alpha_5(1)\mu_5} p_5(0,2) + \frac{2}{\alpha_5(2)\mu_5} p_5(1,2) \\ &\quad + \frac{3}{\alpha_5(3)\mu_5} p_5(1,3) \\ &= \left(\frac{1}{1 \times 2.5} \times 0.5225 \right) \\ &\quad + \left(\frac{1}{2 \times 2.5} \times 0.3962 \right) \\ &\quad + \left(\frac{1}{2 \times 2.5} \times 0.0813 \right) = 0.4163 \end{aligned}$$

Again, we use Equation (2) to compute the overall throughput. We have

$$X(3) = \frac{3}{\sum_{i=1}^5 V_i R_i(1)} = \frac{3}{(1 \times 0.5) + (0.2 \times 1.3120) + (0.3 \times 0.6162) + (0.5 \times 0.2976) + (1 \times 0.4163)} = 1.9816.$$

$$X(2) = \frac{2}{\sum_{i=1}^4 V_i R_i(1)} = \frac{2}{(1 \times 0.5) + (0.2 \times 1.1455) + (0.3 \times 0.5546) + (0.5 \times 0.2727) + (1 \times 0.4)} = 1.3968.$$

At this point we may now compute $G(3)$. We have

$$G(3) = \frac{G(2)}{X(3)} = \frac{0.9844}{1.9816} = 0.4968$$

The final calculation for this first round is to use Equation (3) to determine the mean customer population at each node. We have

$$L_1(1) = X(1)V_1R_1(1) = 0.7273 \times 1.0 \times 0.5 = 0.3636$$

$$L_2(1) = X(1)V_2R_2(1) = 0.7273 \times 0.2 \times 1.0 = 0.1455$$

$$L_3(1) = X(1)V_3R_3(1) = 0.7273 \times 0.3 \times 0.5 = 0.1091$$

$$L_4(1) = X(1)V_4R_4(1) = 0.7273 \times 0.5 \times 0.25 = 0.0909$$

$$L_5(1) = X(1)V_5R_5(1) = 0.7273 \times 1.0 \times 0.4 = 0.2909$$

Before finishing with $k = 1$, we must first compute the marginal for node 5. In particular we need $p_5(0,1)$ and $p_5(1,1)$ We first compute $p_5(1,1)$ from Equation (4). We have

$$p_5(1,1) = \frac{X_5(1)}{\alpha_5(1)\mu_5(1)} p_5(0,0) = \frac{0.7273}{2.5} = 0.2909$$

and, from $p_5(0,1) = 1 - p_5(1,1)$, we have $p_5(0,1) = 0.7091$. We are now ready to begin the second round, this time with $k = 2$. We first compute the mean response time for all five nodes:

$$R_1(2) = \mu_1^{-1} = 0.5$$

$$R_2(2) = \mu_2^{-1}(L_2(1) + 1) = \mu_2^{-1}(0.1455 + 1) = 1.1455$$

$$R_3(2) = \mu_3^{-1}(L_3(1) + 1) = \mu_3^{-1}(0.1091 + 1) = 0.5546$$

$$R_4(2) = \mu_4^{-1}(L_4(1) + 1) = \mu_4^{-1}(0.0909 + 1) = 0.2727$$

$$R_5(2) = \frac{1}{\alpha_5(1)\mu_5} p_5(0,1) + \frac{2}{\alpha_5(2)\mu_5} p_5(1,1) = \left(\frac{1}{1 \times 2.5} \times 0.7091\right) + \left(\frac{1}{2 \times 2.5} \times 0.2909\right) = 0.4$$

Again, we use Equation (2) to compute the overall throughput. We have

At this point we may now compute $G(2)$. We have

$$G(2) = \frac{G(1)}{X(2)} = \frac{1.375}{1.3968} = 0.9844$$

The final calculation for this second round is to use Equation (3) to determine the mean population size at each node. We have

$$L_1(2) = X(2)V_1R_1(2) = 1.3968 \times 1.0 \times 0.5 = 0.6984$$

$$L_2(2) = X(2)V_2R_2(2) = 1.3968 \times 0.2 \times 1.1455 = 0.3200$$

$$L_3(2) = X(2)V_3R_3(2) = 1.3968 \times 0.3 \times 0.577 = 0.2324$$

$$L_4(2) = X(2)V_4R_4(2) = 1.3968 \times 0.5 \times 0.2727 = 0.1905$$

$$L_5(2) = X(2)V_5R_5(2) = 1.3968 \times 1.0 \times 0.4000 = 0.5587$$

The marginals at node 5 may now be computed. We have, from Equation (4),

$$p_5(1,2) = \frac{X_5(2)}{\alpha_5(1)\mu_5} p_5(0,1) = \frac{1.3968}{1 \times 2.5} \times 0.7091 = 0.3962$$

$$p_5(2,2) = \frac{X_5(2)}{\alpha_5(2)\mu_5} p_5(1,1) = \frac{1.3968}{2 \times 2.5} \times 0.2909 = 0.0813$$

and hence $p_5(0,2) = 1 - p_5(1,2) - p_5(2,2) = 0.5225$. We shall do one more step, now augmenting k to 3. Again, the first quantities to compute, from Equation (1), are the mean response times. We obtain

$$R_1(3) = \mu_1^{-1} = 0.5$$

$$R_2(3) = \mu_2^{-1}(L_2(2) + 1) = \mu_2^{-1}(0.3200 + 1) = 1.3200$$

$$R_3(3) = \mu_3^{-1}(L_3(2) + 1) = \mu_3^{-1}(0.2324 + 1) = 0.6162$$

$$R_4(3) = \mu_4^{-1}(L_4(2) + 1) = \mu_4^{-1}(0.1905 + 1) = 0.2976$$

$$R_5(3) = \frac{1}{\alpha_5(1)\mu_5} p_5(0,2) + \frac{2}{\alpha_5(2)\mu_5} p_5(1,2) + \frac{3}{\alpha_5(3)\mu_5} p_5(1,3) = \left(\frac{1}{1 \times 2.5} \times 0.5225\right) + \left(\frac{1}{2 \times 2.5} \times 0.3962\right) + \left(\frac{1}{2 \times 2.5} \times 0.0813\right) = 0.4163$$

Again, we use Equation (2) to compute the overall throughput. We have

$$X(3) = \frac{3}{\sum_{i=1}^5 V_i R_i(1)}$$

$$= \frac{3}{(1 \times 0.5) + (0.2 \times 1.3120) + (0.3 \times 0.6162) + (0.5 \times 0.2976) + (1 \times 0.4163)}$$

Page | 3959 = 1.9816.

At this point we may now compute $G(3)$. We have

$$G(3) = \frac{G(2)}{X(3)} = \frac{0.9844}{1.9816} = 0.4968$$

The final calculation for this second round is to use Equation (3) to determine the mean population size at each node. We have

$$L_1(3) = X(2)V_1R_1(3) = 1.9816 \times 1.0 \times 0.5 = 0.9908$$

$$L_2(3) = X(2)V_2R_2(3) = 1.9816 \times 0.2 \times 1.3200 = 0.5231$$

$$L_3(3) = X(2)V_3R_3(3) = 1.9816 \times 0.3 \times 0.6162 = 0.3663$$

$$L_4(3) = X(1)V_4R_4(3) = 1.9816 \times 0.5 \times 0.2976 = 0.2949$$

$$L_5(3) = X(1)V_5R_5(3) = 1.9816 \times 1.0 \times 0.4163 = 0.8249$$

The marginals at node 5 may now be computed. We have, from Equation (4),

$$p_5(1,3) = \frac{X_5(3)}{\alpha_5(1)\mu_5} p_5(0,2) = \frac{1.9816}{1 \times 2.5} \times 0.5225 = 0.4142$$

$$p_5(2,3) = \frac{X_5(3)}{\alpha_5(2)\mu_5} p_5(1,2) = \frac{1.9816}{2 \times 2.5} \times 0.3692 = 0.1570$$

$$p_5(3,3) = \frac{X_5(3)}{\alpha_5(2)\mu_5} p_5(2,2) = \frac{1.9816}{2 \times 2.5} \times 0.0813 = 0.0322$$

$$p_5(0,3) = 1.0 - p_5(1,3) - p_5(2,3) - p_5(3,3) = 0.3966$$

And so, the process continues in this fashion until the network contains the desired number of customers.

Table 1: Performance Measures for $K = 3$ for Central Server Queueing Model

Service center	1	2	3	4
Mean number of customers $L_i(3)$	1.3055	0.7608	0.5203	0.4136
Mean response time $R_i(3)$	0.5000	1.4569	0.6640	0.3168
Throughput $X_i(3)$	2.6110	0.5222	0.7833	1.3055
Normalization constant $G(3)$	0.1970			

Table 2: Performance Measures for $K = 10$ for Central Server Queueing Model

Service center	1	2	3	4
Mean number of customers $L_i(10)$	2.3347	4.4554	1.9305	1.2794
Mean response time $R_i(10)$	0.5000	4.7709	1.3782	0.5480
Throughput $X_i(10)$	4.6693	0.9339	1.4008	2.3347
Normalization constant $G(10)$	1.0820657E - 5			

Table 3: Performance Measures For $K = 10$ for Modified Load Dependent Server

Service center	1	2	3	4	5
Mean number of customers $L_i(10)$	2.0122	2.6327	1.3338	0.9391	3.0823
Mean response time $R_i(10)$	0.5000	3.2710	1.1047	0.4667	0.7659
Throughput $X_i(10)$	4.0243	0.8049	1.2073	2.0122	4.0244
Normalization constant $G(10)$	1.0724123E - 5				

Table 4: Marginal Distribution at Node 5 with $K = 10$ for Modified Load Dependent Server

N	0	1	2	3	4	5	6	7	8	9	10
$P_5(n)$	0.1009	0.1885	0.1719	0.1519	0.1283	0.1018	0.0739	0.0470	0.0246	0.0093	0.0019

Discussion

Tables I and II show the results obtained after one more step $k = 3$ and with $k = 10$. It is interesting to observe that the size of the normalization constant becomes smaller with large population sizes and we find that $G(20) = 1.3462E-12$ and $G(50) = 1.4631E-33$. While Table III shows the results obtained once we arrive at $k = 10$. The marginal distribution of customers at node 5,

when the network contains $k = 10$ customers is shown in Table IV

Conclusion

A closed queueing network where a fixed number of customers forever cycles among the service centers, never to depart has been investigated, in order to provide some insight into the performance measures in closed queue networks. Our quest is to exempt the normalization constant in the computation of performance measure in the closed queueing network. The arrival properties and Little's law are use with the help of some existing equations and formulas in queueing network. Performance measures, such as Mean number of customer, response time, throughput, and marginal probability distribution are obtained for central server and load dependent server closed queueing networks for node 4 and node, 5 for $k = 3$ and $k = 10$ as the results are shown in Tables I – IV respectively.

Conflict of interest

Authors declare no conflict of interest whatsoever.

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