



Partial Fraction Expansion of Meromorphic Maps

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Page | 4284

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ABSTRACT

Partial fraction expansion of a meromorphic function or map is a way of expressing it as an infinite series of rational functions and polynomials. This representation is particularly useful in the summation of series in mathematical analysis and has at its heart the Mittag-Leffler's expansion theorem which holds for arbitrary domains. In this article, the Mittag-Leffler's expansion theorem was used to obtain partial fraction expansions of the functions $\tan z$ and $\coth z$ which are meromorphic in the entire complex plane. The expansion of $\tan z$ was then used to obtain the Laurent series for it where except for the odd coefficients, all the even coefficients in the series were shown to vanish identically. By comparing these coefficients with those of the well-known Maclaurin series for $\tan z$ the exact sum of the odd terms in the important p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 2, 4, 6$) were obtained in closed form and consequently, those of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 2, 4, 6$) for all values of n . The partial fraction expansion of $\coth z$ was also gainfully employed in finding the exact sum (in closed form) of the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+z^2}$ ($z \neq ni, n = 0, 1, 2, \dots$) to also show its usefulness in application. Finally, since closed form expressions were obtained for those series found from the partial fraction expansions of the functions considered in this article, we suggest that further research in this area should focus on the application of the Mittag-Leffler's theorem to other meromorphic functions such as $\sec z$, $\cot z$ etc to find similar results on the summation of series as done here.

Keywords: Meromorphic function, Residue, Cauchy's residue theorem, Partial fraction expansion of meromorphic functions, Mittag-Leffler's expansion theorem, Laurent series.

Introduction

A function or map $f(z)$ which is analytic in a domain D except for poles is said to be meromorphic there. The maps $\tan z$ and $\coth z$ are meromorphic in the whole of the z plane with simple poles at $z = \left(n + \frac{1}{2}\right)\pi$ ($n = 0, \pm 1, \pm 2, \dots$) and $z = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$), respectively. These maps and indeed all maps that are

meromorphic in the whole of the z plane can be expressed as the quotient of two entire functions; that is, functions which are analytic in the whole of the z plane. This important result is a corollary of Weierstrass factorization theorem for entire functions as infinite products and can be found in Ahlfors (1979). It therefore follows that

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meromorphic functions can also be expressed as infinite products as is evident in Ahlfors (1979) and Chaitanya's (2011) random pages. We note here too that a function $f(z)$ which is meromorphic in a domain D which is not the whole of the complex plane is also the quotient $f(z) = h(z)/g(z)$ of two analytic functions $h(z)$ and $g(z)$ in D , provided that $g(z)$ is not identically zero. The poles of $f(z)$ are then the zeros of $g(z)$, although a common zero of $h(z)$ and $g(z)$ can also be a removable singularity (Ahlfors, 1979). Another equally useful representation is by use of partial fraction expansion which is a way of expressing a meromorphic function as an infinite series of rational functions and polynomials and is the concern of this paper. This representation is very useful in the summation of series and always arises in the course of application of Weierstrass theorem for product representation of entire functions having infinitely many zeros $z_n (n = 1, 2, 3, \dots)$ with $\lim_{n \rightarrow \infty} z_n = \infty$ (Ahlfors 1979; Swem and Swem, 2020). This paper will only focus on the former application and use the approach that is based on the most general theorem of Mittag-Leffler's which holds for arbitrary domains to express $\tan z$ and $\coth z$ as partial fractions. These expansions will then be used in finding, respectively, the precise sums of the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 2, 4, 6$), the odd terms in the p -series for the same cases, and $\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + z^2}$ ($z \neq ni, n = 0, 1, 2, \dots$). Cauchy's residue theorem is at the heart of the Mittag-Leffler's theorem and states that if a function $f(z)$ is analytic interior to and on a simple closed curve C except at a finite number of singularities a_1, a_2, a_3, \dots inside C with respective residues of $f(z)$ as $a_{-1}, a_{-2}, a_{-3}, \dots$, then

$$\oint_C f(z) dz = 2\pi i \sum_n a_{-n} \quad (1)$$

In order to identify the type of singularity $z = z_0$ of an analytic function $f(z)$ and its residue there requires the Laurent expansion theorem. This

fundamental theorem in the theory of analytic functions states that if a function $f(z)$ is analytic within an annulus $D = \{z: R_1 < |z - z_0| < R_2\}$ and on its boundary consisting of two concentric circles $C_1 = \{z: |z - z_0| = R_1\}$ and $C_2 = \{z: |z - z_0| = R_2\}$, then at every point $z \in D$, $f(z)$ has the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad (2)$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z) dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \dots) \quad (3)$$

and

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z) dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, 3, \dots) \quad (4)$$

(Spiegel, 1974; Churchill and Brown, 1984). The part $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called the analytic part of the Laurent series while the remaining part $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$ consisting of reciprocal powers of $(z - z_0)$ is called the principal part. An important consequence of Cauchy's integral theorem that is used to neatly combine the coefficients in equations (3) and (4) of the exponents of $(z - z_0)$ in equation (1) states that if a function $f(z)$ is analytic in a domain D which is bounded by two simple closed curves Γ_1 and Γ_2 where Γ_1 lies inside Γ_2 , then for each $z \in D$

$$\oint_{\Gamma_2} f(z) dz = \oint_{\Gamma_1} f(z) dz \quad (5)$$

where Γ_1 and Γ_2 are both traversed in the positive sense relative to their interiors (Ahlfors, 1979; Spiegel, 1974; Churchill and Brown, 1984). Noting that the integrands $f(z)/(z - z_0)^{n+1}$ and $f(z)/(z - z_0)^{-n+1}$ in equations (3) and (4) are both analytic inside the annulus and on its boundary, we then replace the paths of integration C_1 and C_2 by the circle $C = \{z: |z - z_0| = R\}$ where $R_1 < R < R_2$ and apply the preceding result to obtain

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \dots) \quad (6)$$

Also, if $f(z)$ is analytic everywhere inside and on the circle C_2 except at the origin z_0 , we then take the radius R_1 of C_1 to be arbitrarily small. Expansion (2) is then valid for $0 < |z - z_0| < R_2$. The residue of $f(z)$ at the singular point $z = z_0$, denoted by $\text{Res}[f(z)]_{z=z_0}$, is by definition the coefficient a_{-1} of the term $1/(z - z_0)$ in series (2) and has the integral representation

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz \quad (7)$$

obtained by setting $n = -1$ in equation (6). The type of singularity is determined from the form of the Laurent series (2). In particular, for poles, the Laurent series takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^k \frac{a_{-n}}{(z - z_0)^n} \quad (8)$$

where $z = z_0$ is called a pole of order k . When $k = 1$ the pole is called simple. Using the expression in (8) the residue of $f(z)$ at a pole of order k is given by the formula (Spiegel, 1974; Churchill and Brown, 1984) as

$$a_{-1} = \frac{1}{(k-1)!} \lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} \{(z - z_0)^k f(z)\} \quad (9)$$

When $z = z_0$ is a simple pole formula (9) reduces to

$$a_{-1} = \lim_{z \rightarrow z_0} (z - z_0) f(z) \quad (10)$$

There is indeed a close relationship between the partial fraction representation of a meromorphic function and its Laurent series. In fact, the partial fraction representation of a meromorphic function can also be used to find the Laurent series for it by simply replacing the rational functions in the sum with their Laurent series. This can lead to very interesting identities and aid in the summation of infinite series if a Laurent series is already known (Markushevich, 1965)

Finally, we note here that while Mittag-Leffler's expansion theorem holds for arbitrary domains, a particular form of the theorem that is applicable to

functions which are meromorphic in the entire complex plane only can be found in Ahlfors (1979), Conway (1978), Slizi and Tóth (2012), Turner (2013), Mittag-Leffler's theorem in Encyclopedia of Mathematics (2020). In this approach, it is proven that given an infinite sequence $\{a_n\}$ of nonzero complex numbers with $\lim_{n \rightarrow \infty} a_n = \infty$ and if $P_n(\zeta)$ are polynomials without constant term, then there are functions which are meromorphic in the entire complex plane with poles precisely at the points a_n and the corresponding principal parts $P_n\left(\frac{1}{z - a_n}\right)$. The most general meromorphic function of this kind can then be expressed in the form

$$f(z) = \sum_n \left[P_n\left(\frac{1}{z - a_n}\right) - p_n(z) \right] + g(z) \quad (11)$$

where the $p_n(z)$ are suitably chosen polynomials and $g(z)$ is an entire function.

Materials and Methods

We now state and prove the Mittag-Leffler's expansion theorem following the approach of Spiegel (1974).

Mittag-Leffler's expansion theorem

Suppose that the only singular points of the function $f(z)$ in the finite complex plane are the simple poles a_1, a_2, a_3, \dots arranged such that the order relation $|a_1| \leq |a_2| \leq |a_3| \leq \dots$ is satisfied. Let the residues of $f(z)$ at a_1, a_2, a_3, \dots be $a_{-1}, a_{-2}, a_{-3}, \dots$, respectively. If C_N be circles with radius R_N which do not pass through any poles and on which $|f(z)| < M$, where M is independent of N and $R_N \rightarrow \infty$ as $N \rightarrow \infty$, then

$$f(z) = f(0) + \sum_{n=1}^{\infty} a_{-n} \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right) \quad (12)$$

Proof

Let $z = a_n$ ($n = 1, 2, 3, \dots$) be the poles of $f(z)$ and suppose that $f(z)$ has no pole at $z = \zeta$. Consider the map $\frac{f(z)}{z-\zeta}$ which evidently has poles of order one (that is, simple poles) at $z = a_n$ ($n = 1, 2, 3, \dots$) and ζ . Now, by formula (10)

$$\operatorname{Res}_{z=a_n} \left[\frac{f(z)}{z-\zeta} \right] = \lim_{z \rightarrow a_n} (z - a_n) \frac{f(z)}{z-\zeta} = \frac{a-n}{a_n-\zeta}$$

$$\operatorname{Res}_{z=\zeta} \left[\frac{f(z)}{z-\zeta} \right] = \lim_{z \rightarrow \zeta} (z - \zeta) \frac{f(z)}{z-\zeta} = f(\zeta)$$

On applying Cauchy's residue theorem yields

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z-\zeta} dz = f(\zeta) + \sum_n \frac{a-n}{a_n-\zeta} \quad (13)$$

Suppose that $f(z)$ is also analytic at $z = 0$. It then follows that $f(z)$ is defined and single-valued at $z = 0$ and at each point in some δ -neighbourhood of the point zero. Then setting $\zeta = 0$ in equation (13) yields

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z} dz = f(0) + \sum_n \frac{a-n}{a_n} \quad (14)$$

where the circle C_N has the equation $z = R_N e^{i\theta}$ ($0 \leq \theta < 2\pi$) or $|z| = R_N$.

On subtracting equation (14) from (13), we have

$$\frac{\zeta}{2\pi i} \oint_{C_N} \frac{f(z)}{z(z-\zeta)} dz = f(\zeta) - f(0) + \sum_n a_{-n} \left(\frac{1}{a_n-\zeta} - \frac{1}{a_n} \right) \quad (15)$$

Since the point ζ is interior to C_N , it follows that for any point z on C_N

$$|z - \zeta| \geq ||z| - |\zeta|| = |z| - |\zeta| = |R_N| - |\zeta| \quad (16)$$

It is well known in complex analysis that if a function $g(z)$ is integrable along a rectifiable curve C of length L and if $|g(z)| \leq M$ on C , then

$$\left| \int_C g(z) dz \right| \leq ML$$

(Spiegel, 1974). Using this result and the inequality in (16), we obtain the estimate

$$\left| \oint_{C_N} \frac{f(z)}{z(z-\zeta)} dz \right| \leq \frac{2\pi M}{R_N - |\zeta|} \quad (17)$$

where C_N has length $2\pi R_N$. As $N \rightarrow \infty$ and hence $R_N \rightarrow \infty$, the right-hand side of inequality (17) tends to zero, and we have that

$$\lim_{N \rightarrow \infty} \oint_{C_N} \frac{f(z)}{z(z-\zeta)} dz = 0$$

On letting $N \rightarrow \infty$ in equation (15) and replacing ζ by z yields the required result.

Results

The Mittag-Leffler's theorem is now applied in finding the partial fraction expansions of the meromorphic functions $\tan z$ and $\coth z$ and the resulting expansions employed in the summation of some important infinite series. We begin with the partial fraction expansion of $\tan z$ and consider:

Problem 1. To prove that $\tan z$ has the partial fraction expansion given as

$$\tan z = \sum_{n=0}^{\infty} \frac{-2z}{z^2 - \left(n + \frac{1}{2}\right)^2 \pi^2} \quad (18)$$

As earlier stated, the function $\tan z$ has simple poles which are all real at the points $z = a_n = \left(n + \frac{1}{2}\right)\pi$ ($n = 0, \pm 1, \pm 2, \dots$). These singularities can be arranged in the order of increasing absolute value as $\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \dots$. The residue of $\tan z$ at each of the poles is

$$a_{-n} = \lim_{z \rightarrow \left(n + \frac{1}{2}\right)\pi} \left[z - \left(n + \frac{1}{2}\right)\pi \right] \tan z = -1 \quad (19)$$

for every $n \in \mathbb{N}$, where \mathbb{N} is the set of natural numbers. Noting that the poles of $\tan z$ can also be written in the form $z = \left(n - \frac{1}{2}\right)\pi$ ($n = 0, \pm 1, \pm 2, \dots$) on their re-arrangement as $-\frac{\pi}{2}, \frac{\pi}{2}, -\frac{3\pi}{2}, \frac{3\pi}{2}, \dots$, we choose the circle $c_N: z = N\pi e^{i\theta}$ ($0 \leq \theta < 2\pi$) where $N \in \mathbb{N}$, which clearly does not pass through any of the poles and encloses the first $2N$ poles of equal numerical magnitude but

opposite in sign. The function $\tan z$ is bounded on this circle whose radius $N\pi = R_N \rightarrow \infty$ as $N \rightarrow \infty$ as required by the Mittag-Leffler's theorem. We prove this by obtaining an upper bound for $|\tan z|$ independent of N on the parts of c_N which lie in the upper half $\text{Im } z > 0$ of the z plane, the lower half $\text{Im } z < 0$ of the z plane, and at the points $z = N\pi$ and $z = -N\pi$ corresponding to the circle's x intercepts. Now

$$|\tan z| = \left| \frac{1}{i} \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \right| \leq \frac{|e^{iz}| + |e^{-iz}|}{||e^{iz}| - |e^{-iz}||} = \frac{e^{-N\pi \sin \theta} + e^{N\pi \sin \theta}}{e^{-N\pi \sin \theta} - e^{N\pi \sin \theta}} \quad (20)$$

Case 1: Part of the circle c_N which lie in the upper half $\text{Im } z > 0$ of the z plane

In this case $\sin \theta > 0$ for all values of $\theta \in (0, \pi)$ and the inequality in (20) simplifies to

$$|\tan z| \leq \frac{e^{N\pi \sin \theta} + e^{-N\pi \sin \theta}}{e^{N\pi \sin \theta} - e^{-N\pi \sin \theta}} = \frac{1 + e^{-2N\pi \sin \theta}}{1 - e^{-2N\pi \sin \theta}} \quad (21)$$

Since $R_N = N\pi > \frac{1}{2}$ for every natural number N and $\sin \theta > 0$, we obtain from inequality (21) an upper bound for $|\tan z|$ in terms of θ as

$$|\tan z| < \frac{1 + e^{-\sin \theta}}{1 - e^{-\sin \theta}} < \frac{2}{1 - e^{-\sin \theta}} \quad (22)$$

which holds for every $\theta \in (0, \pi)$. In particular, when $\theta = \frac{\pi}{2}$, the number $2/(1 - e^{-1})$ is an upper bound for $\tan z$ on c_N .

Case 2: Part of the circle c_N which lie on the lower half $\text{Im } z < 0$ of the z plane

In this case $\sin \theta < 0$ for all values of $\theta \in (\pi, 2\pi)$ and the inequality in (20) simplifies to

$$|\tan z| \leq \frac{e^{-N\pi \sin \theta} + e^{N\pi \sin \theta}}{e^{-N\pi \sin \theta} - e^{N\pi \sin \theta}} = \frac{1 + e^{2N\pi \sin \theta}}{1 - e^{2N\pi \sin \theta}} \quad (23)$$

Again, since $R_N = N\pi > \frac{1}{2}$ for every natural number N , we obtain from inequality (23) an upper bound for $|\tan z|$ as

$$|\tan z| < \frac{1 + e^{\sin \theta}}{1 - e^{\sin \theta}} = \frac{1 + e^{-\sin \alpha}}{1 - e^{-\sin \alpha}} < \frac{2}{1 - e^{-\sin \alpha}} \quad (24)$$

where $\alpha \in (0, \pi)$.

Case 3: Point $z = N\pi$. For this case $|\tan N\pi| = 0$

Case 4: Point $z = -N\pi$. For this case too $|\tan(-N\pi)| = 0$

From cases (1) - (4), it is clear that $\tan z$ is bounded on the circle c_N by the number $\frac{2}{1 - e^{-\sin \theta}}$ for every value of $\theta \in (0, \pi)$. Thus, by Mittag-Leffler's theorem

$$\begin{aligned} \tan z &= \sum_n -1 \left\{ \frac{1}{z - (n + \frac{1}{2})\pi} + \frac{1}{(n + \frac{1}{2})\pi} \right\} \\ &= -\lim_{N \rightarrow \infty} \left[\sum_{n=-N}^{-1} \left(\frac{1}{z - (n + \frac{1}{2})\pi} + \frac{1}{(n + \frac{1}{2})\pi} \right) + \sum_{n=0}^{N-1} \left(\frac{1}{z - (n + \frac{1}{2})\pi} + \frac{1}{(n + \frac{1}{2})\pi} \right) \right] \\ &= -\lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \frac{2z}{z^2 - (n + \frac{1}{2})^2 \pi^2} \\ &= \sum_{n=0}^{\infty} \frac{-2z}{z^2 - (n + \frac{1}{2})^2 \pi^2} \quad \text{as required.} \end{aligned}$$

We now use the partial fraction expansion of $\tan z$ in equation (18) to find its Laurent series at $z = 0$ and then apply the result obtained in summing the infinite series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k+2}}$ where $k = 0, 1, 2, \dots$. We first note that the expression in the summation sign on the right hand side of equation (18) can be expanded using a geometric series as follows:

$$\begin{aligned} \frac{-2z}{z^2 - (n + \frac{1}{2})^2 \pi^2} &= \frac{8z}{(2n+1)^2 \pi^2} \cdot \frac{1}{1 - \left[\frac{2z}{(2n+1)\pi} \right]^2} = \\ &= \frac{8z}{(2n+1)^2 \pi^2} \sum_{k=0}^{\infty} \left(\left[\frac{2z}{(2n+1)\pi} \right]^2 \right)^k \quad (25) \end{aligned}$$

Substituting in equation (18) yields the Laurent series for $\tan z$ as

$$\tan z = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{2k+3}}{(2n+1)^{2k+2} \pi^{2k+2}} z^{2k+1} \quad (26)$$

Equation (26) is important and shows that the coefficients in the Laurent series for $\tan z$ are given as

$$a_{2k+1} = \frac{2^{2k+3}}{\pi^{2k+2}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k+2}} \quad (27) \text{ and}$$

$$a_{2k} = 0 \quad (28)$$

By comparing the coefficients in equation (27) to the power series expansion of $\tan z$ at $z = 0$ (that is, the Maclaurin series expansion of $\tan z$), namely,

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots$$

yields the sum of the following infinite series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \quad (29)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \quad (30)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad (31)$$

and so on. Observe that each of the infinite series in equations (29) to (31) consist of some even power of the odd terms in the important p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (converges when $p > 1$ and diverges when $p \leq 1$) for the cases $p = 2, 4$, and 6 , respectively, and hence they all converge by comparison with the corresponding p -series. On writing

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

we obtain the sum of the p -series for the case $p = 2$ as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (32)$$

Similarly, from equations (30) and (31) we obtain the sums of the p -series for cases $p = 4$ and 6 respectively, as

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad (33)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \quad (34)$$

Problem 2. Finally, we prove that

$$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2 \pi^2} \quad (35)$$

Here we consider the function $f(z) = \coth z - \frac{1}{z}$ which has a removable singularity at $z = 0$ since

$$\lim_{z \rightarrow 0} \left(\coth z - \frac{1}{z} \right) = \lim_{z \rightarrow 0} \left(\frac{z \cosh z - \sinh z}{z \sinh z} \right) = 0$$

We therefore define $f(0) = 0$. The function $f(z) = \coth z - \frac{1}{z}$ has simple poles at $z = n\pi i$ ($n = \pm 1, \pm 2, \dots$) which can be arranged in order of increasing absolute value as $+\pi i, -\pi i, +2\pi i, -2\pi i, \dots$. The residue of $f(z)$ at these poles is

$$a_{-n} = \lim_{z \rightarrow n\pi i} (z - n\pi i) \left(\coth z - \frac{1}{z} \right) = 1$$

Consider the circles $c_N: z = \left(N + \frac{1}{2}\right) \pi e^{i\theta}$ ($0 \leq \theta < 2\pi$) where $N \in \mathbb{N}$, which evidently do not pass through any of the poles of $f(z) = \coth z - \frac{1}{z}$. On both parts of c_N which lie in the left and right halves of the z plane $\operatorname{Re} z < 0$ and $\operatorname{Re} z > 0$, $\left| \coth z - \frac{1}{z} \right| < \frac{2}{1 - e^{-\cos \theta}}$ ($-\frac{\pi}{2} < \theta < \frac{\pi}{2}$) while $\left| \coth z - \frac{1}{z} \right| < 1$ at the remaining points $-\left(N + \frac{1}{2}\right) \pi i$ and $\left(N + \frac{1}{2}\right) \pi i$ on c_N . Since the real number $\frac{2}{1 - e^{-\cos \theta}} > 1$ for every value of $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows that any value of $\frac{2}{1 - e^{-\cos \theta}}$ is an upper bound for $\coth z - \frac{1}{z}$ independent of N on c_N .

Thus, by the Mittag-Leffler's theorem

$$\coth z - \frac{1}{z} = \sum_{n \neq 0} \left(\frac{1}{z-n\pi i} + \frac{1}{n\pi i} \right)$$

$$= \lim_{N \rightarrow \infty} \left\{ \sum_{n=-N}^{-1} \left(\frac{1}{z-n\pi i} + \frac{1}{n\pi i} \right) + \sum_{n=1}^N \left(\frac{1}{z-n\pi i} + \frac{1}{n\pi i} \right) \right\}$$

$$= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{2z}{z^2 + n^2 \pi^2}$$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2 \pi^2}$$

Hence,

$$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2 \pi^2}$$

We now use the partial fraction expansion of $\coth z$ in equation (35) to show that the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (36)$$

On replacing z by πz in equation (35) and simplifying yields

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2} = \frac{\pi}{2z} \coth \pi z - \frac{1}{2z^2} \quad (37)$$

On taking limits as $z \rightarrow 0$ on both sides of equation (37) and applying L'Hospital's rule yields

$$\lim_{z \rightarrow 0} \sum_{n=1}^{\infty} \frac{1}{z^2 + n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{z \rightarrow 0} \left(\frac{\pi z \coth \pi z - 1}{2z^2} \right) = \frac{\pi^2}{6}$$

as required. Observe that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

as already obtained in equation (29). Also, since

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2} = \sum_{n=-\infty}^{-1} \frac{1}{n^2 + z^2}$$

it follows that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + z^2} = \sum_{n=-\infty}^{-1} \frac{1}{n^2 + z^2} + \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{n^2 + z^2} = \frac{\pi}{z} \coth \pi z \quad (38)$$

Discussion

We begin the discussion with equation (18) for the partial fraction expansion of $\tan z$. This equation is of interest since it is closely related to its Taylor series at $z = 0$ as shown in equation (26) from which the coefficients of the Taylor series were determined. The general formulae for the odd and even coefficients, equations (27) and (28) respectively, show that only the odd terms in the Taylor series for $\tan z$ are present in the expansion. Using equation (27) for the coefficients of the odd powers of z and the well known Taylor series expansion for $\tan z$ at $z = 0$, the precise sum (in closed form) of the even powers of the odd terms in the p -series, namely $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k+2}}$ for the cases $k = 0, 1$, and 2 were determined and shown in equations (29), (30), and (31), respectively. Also by direct manipulation of the series in equations (29), (30), and (31), respectively, the values of the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for the cases $p = 2, 4$, and 6 were also determined as shown in equations (32), (33), and (34), respectively. The most interesting case for which $p = 2$ (the once famous Basel problem) was first found by Leonhard Euler in 1734, and read on the 5th of December, 1735 in the Saint Petersburg Academy of Science (Derbyshire, 2003). Euler also gave a rigorous proof that the correct sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (in closed form) is indeed $\frac{\pi^2}{6}$.

Just like in the case of $\tan z$, the partial fraction expansion of $\coth z$ was also shown to have applications in the summation of series. Here too the partial fraction expansion of $\coth z$ was used to find the exact sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ in the Basel problem. Furthermore, by direct manipulation of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ the sum of the

infinite series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ in equation (29) was also obtained in closed form as $\frac{\pi^2}{8}$. Finally, the partial fraction expansion of $\coth z$ was also used to show that the infinite series $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+z^2}$ has exact sum (in closed form) equal to $\frac{\pi}{z} \coth \pi z$ provided $z \neq ni$ ($n = 0, 1, 2, \dots$).

Conclusion

As shown in this paper, the Mittag-Leffler's expansion theorem was used to find the partial fraction expansions of $\tan z$ and $\coth z$ and the results gainfully employed in the summation of some important series in Mathematical analysis. In closing, we remark that the Mittag-Leffler's expansion theorem is a powerful result that is applicable to arbitrary domains and at the same time extremely easy to apply. We therefore recommend that it should also be applied to obtain partial fraction expansions of other meromorphic functions such as $\sec z$, $\cot z$ etc and to obtain similar results on the summation of other series as done in this paper.

Conflict of Interest

The author declares no conflict of interest concerning the publication of this article.

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